3
Quantum states of bosonic systems

The radiation field is an example of an immaterial bosonic system. Examples of material bosonic systems which play an important role in quantum optics are the vibrations of trapped atoms and the internuclear vibrations of molecules inasmuch as they can be approximated by harmonic oscillators. Among the wide variety of possible quantum states of bosonic systems, there are some fundamental types that are of particular interest. First, such states may serve as a quantum-mechanical basis for representing the various observables of interest. Second, they may be regarded as typical examples for certain limiting cases of quantum noise, with special emphasis on nonclassical features. Although the definition of nonclassicality (Chapter 8) is a nontrivial task, we will use the term nonclassical in this chapter in a more generous way.

We will begin with the introduction of the number states (Section 3.1). Although they are well known, from standard quantum-mechanics textbooks, as energy eigenstates of harmonic oscillators, they will be seen to reveal the quite counter-intuitive feature of not showing the oscillatory behavior expected of classical harmonic oscillators. The classically expected oscillatory behavior is then shown to be realized by the coherent states (Section 3.2). The coherent states are Gaussian states to which the squeezed coherent states (often called quadrature-squeezed states) also belong (Section 3.3). Their quantum-noise properties are of great practical relevance for applications in measurement techniques below the standard quantum limit. Finally, the eigenstates of phase-rotated quadratures (Section 3.4) and phase states (Section 3.5) are introduced. Quadrature eigenstates play an important role in the context of homodyne detection (see Sections 6.5 and 7.1).

3.1
Number states

Let us consider a system of uncoupled harmonic oscillators whose Hamiltonian reads

$$\hat{H} = \sum_{\lambda} \hat{H}_{\lambda} \,, \tag{3.1}$$

Quantum Optics, Third revised and extended edition. Werner Vogel and Dirk-Gunnar Welsch

ISBN: 3-527-40507-0

where

$$\hat{H}_\lambda = \hbar\omega_\lambda\big(\hat{n}_\lambda + \tfrac{1}{2}\big) \tag{3.2}$$

is the single-oscillator Hamiltonian expressed in terms of the number operator

$$\hat{n}_\lambda = \hat{a}_\lambda^\dagger \hat{a}_\lambda \tag{3.3}$$

of the λth oscillator, with $\hat{a}_\lambda$ and $\hat{a}_\lambda^\dagger$, respectively, being the annihilation and creation operators attributed to the oscillator which obey the bosonic commutation relations

$$[\hat{a}_\lambda, \hat{a}_{\lambda'}^\dagger] = \delta_{\lambda\lambda'} \tag{3.4}$$

$$[\hat{a}_\lambda, \hat{a}_{\lambda'}] = 0 = [\hat{a}_\lambda^\dagger, \hat{a}_{\lambda'}^\dagger]. \tag{3.5}$$

From Chapter 2 we know that mode expansion of the free radiation field just leads to Eqs (3.1)–(3.5) [cf. Eqs (2.75)–(2.78)]. Therefore, what we derive in the following in a very general context can be thought of as being the radiation field or other specific realizations of harmonic oscillators – also referred to as modes in the following – such as the (harmonic) motion of trapped atoms or (harmonic) molecular vibrations.

To proceed we will restrict ourselves first to a single oscillator, thereby omitting the index λ ($\hat{H}_\lambda \mapsto \hat{H}$), and we will then later consider the case of multimode systems. The eigenstates of the Hamiltonian (3.2) are of particular interest, since they naturally provide a complete and orthonormal set of basis states. To solve the eigenvalue problem for $\hat{H}$, it is sufficient to solve the eigenvalue problem for the number operator $\hat{n}$, since both operators commute, $[\hat{H}, \hat{n}] = 0$. The main steps for solving this standard problem of quantum mechanics may be summarized as follows. The eigenvalue equation reads

$$\hat{n}|\phi_n\rangle = n|\phi_n\rangle, \tag{3.6}$$

with n being the eigenvalue and $|\phi_n\rangle$ the corresponding eigenvector. Since $\hat{n}$ is Hermitian, the eigenvalues n are real-valued and eigenvectors corresponding to different eigenvalues are orthogonal. Inserting $\hat{n}$ as given according to Eq. (3.3) in

$$\langle\phi_n|\hat{n}|\phi_n\rangle = n\langle\phi_n|\phi_n\rangle, \tag{3.7}$$

we obtain

$$\langle\phi_n|\hat{a}^\dagger\hat{a}|\phi_n\rangle = n\langle\phi_n|\phi_n\rangle. \tag{3.8}$$

Since $|\phi_n\rangle$ and $\hat{a}|\phi_n\rangle$ are Hilbert-space vectors, whose norms are non-negative, we immediately observe from Eq. (3.8) that the eigenvalue n must also be non-negative. Using the relation $[\hat{a}^l, \hat{n}] = l\hat{a}^l$, derived with the help of the bosonic commutator (3.4), we obtain

$$\hat{n}\hat{a}^l|\phi_n\rangle = (\hat{a}^l\hat{n} + [\hat{n}, \hat{a}^l])|\phi_n\rangle = (n-l)\hat{a}^l|\phi_n\rangle. \tag{3.9}$$

From this relation it is seen that, as long as $n - l \geq 0$, the state $\hat{a}^l|\phi_n\rangle$ is an eigenstate of $\hat{n}$ with eigenvalue $n-l$, which may be denoted by $|\phi_{n-l}\rangle = \hat{a}^l|\phi_n\rangle$. For a negative eigenvalue $n-l<0$ it is required that $\hat{a}^l|\phi_n\rangle = 0$ in order to fulfill Eq. (3.9) under the constraint of non-negative eigenvalues. For $n=0$ and $l=1$ this requirement provides us with the relation

$$\hat{a}|\phi_0\rangle = 0, \tag{3.10}$$

which defines $|\phi_0\rangle$ as the ground state, having zero number of quanta, from which it follows that no further quantum can be annihilated. Analogously to Eq. (3.9), we may prove that

$$\hat{n}\hat{a}^{\dagger l}|\phi_n\rangle = (n+l)\hat{a}^{\dagger l}|\phi_n\rangle, \tag{3.11}$$

which states that $\hat{a}^{\dagger l}|\phi_n\rangle$ is an eigenstate of $\hat{n}$ with eigenvalue $n+l$, which we denote by $|\phi_{n+l}\rangle = \hat{a}^{\dagger l}|\phi_n\rangle$.

From Eqs (3.9) and (3.11) it is clear that $\hat{a}$ and $\hat{a}^\dagger$ decrease and increase the number of energy quanta by single quanta, respectively. Therefore, the operators $\hat{a}$ and $\hat{a}^\dagger$ are called annihilation and creation operators, respectively. Starting from the ground state $|\phi_0\rangle$ via Eq. (3.11) a ladder of eigenstates of the free Hamiltonian (3.2) is created by multiple application of the creation operator $\hat{a}^\dagger$. Since these states exhibit defined numbers of energy quanta they are usually called number states. Depending on the physical system under study they may correspond to eigenstates with a precise number of photons, phonons, or other elementary bosonic excitations.

Normalizing the states $|\phi_n\rangle$, we obtain the number states $|n\rangle = |\phi_n\rangle / \langle\phi_n|\phi_n\rangle$ as an orthonormal set of basis states. For this purpose we create the states $|n\rangle$ via Eq. (3.11) from the ground state $|0\rangle$ – also called the vacuum state – which we take to be normalized, $\langle 0|0\rangle = 1$,

$$|n\rangle = \mathcal{N}_n \hat{a}^{\dagger n}|0\rangle. \tag{3.12}$$

The value of $\mathcal{N}_n$ is determined by the normalization condition

$$\langle n|n\rangle = 1, \tag{3.13}$$

which by insertion of Eq. (3.12) into Eq. (3.13) reads

$$|\mathcal{N}_n|^2 \langle 0|\hat{a}^n\hat{a}^{\dagger n}|0\rangle = 1. \tag{3.14}$$

The simplest way of calculating the vacuum expectation value in Eq. (3.14) is to bring the operator product $\hat{a}^n\hat{a}^{\dagger n}$ into normal order. Applying Eq. (C.34) yields

$$\hat{a}^n \hat{a}^{\dagger n} = :\left(\hat{a} + \frac{\partial}{\partial \hat{a}^\dagger}\right)^n \hat{a}^{\dagger n}: = \sum_{l=0}^{n} \binom{n}{l} :\hat{a}^{n-l} \left(\frac{\partial}{\partial \hat{a}^\dagger}\right)^l \hat{a}^{\dagger n}:$$

$$= \sum_{l=0}^{n} \binom{n}{l} \frac{n!}{(n-l)!} \hat{a}^{\dagger n-l} \hat{a}^{n-l}, \tag{3.15}$$

so that we may rewrite Eq. (3.14) as

$$|\mathcal{N}_n|^2 \sum_{l=0}^{n} \binom{n}{l} \frac{n!}{(n-l)!} \langle 0|\hat{a}^{\dagger n-l} \hat{a}^{n-l}|0\rangle = 1. \tag{3.16}$$

Since quanta cannot be annihilated from the vacuum state, cf. Eq. (3.10), we conclude that only the term with $n-l=0$ contributes to the sum of Eq. (3.16), so that

$$\mathcal{N}_n = \frac{1}{\sqrt{n!}}, \tag{3.17}$$

where we have chosen $\mathcal{N}_n$ to be real-valued. Inserting Eq. (3.17) into Eq. (3.12), we obtain a rule for creating from the vacuum state all the number states $|n\rangle$, which are the eigenstates of the Hamiltonian (3.2),

$$|n\rangle = \frac{1}{\sqrt{n!}} \hat{a}^{\dagger n} |0\rangle. \tag{3.18}$$

This implies also the following relations for the (normalized) number states:

$$\hat{a}^\dagger |n\rangle = \frac{1}{\sqrt{n!}} \hat{a}^{\dagger n+1} |0\rangle = \sqrt{n+1}\, |n+1\rangle, \tag{3.19}$$

$$\hat{a}\, |n\rangle = \frac{1}{\sqrt{n!}} \hat{a}\, \hat{a}^{\dagger n} |0\rangle = \frac{1}{\sqrt{n!}} [\hat{a}, \hat{a}^{\dagger n}] |0\rangle$$

$$= \frac{n}{\sqrt{n!}} \hat{a}^{\dagger n-1} |0\rangle = \sqrt{n}\, |n-1\rangle, \tag{3.20}$$

where the relations (C.16) and (3.10) have been used for the derivation of Eq. (3.20).

As already mentioned, the number states $|n\rangle$ and $|m\rangle$ with $n \neq m$ are orthogonal, since they are eigenstates of the Hermitian number operator $\hat{n}$,

$$\langle m|n\rangle = \delta_{mn}. \tag{3.21}$$

This orthogonality may also be shown directly, by representing $|n\rangle$ and $|m\rangle$ in the form (3.18) and normally ordering the operators $\hat{a}$ and $\hat{a}^\dagger$. Moreover, the number states form a complete set of orthonormal vectors in the Hilbert space of the single-mode system,

$$\sum_{n=0}^{\infty} |n\rangle\langle n| = \hat{I}, \tag{3.22}$$

where $\hat{I}$ is the unity operator in this Hilbert space. It is now evident that, due to their orthonormality (3.21) and completeness (3.22), the number states are of potential use for many quantum-mechanical calculations.

3.1.1
Statistics of the number states

Let us consider the main properties of number states, focusing on the free radiation field expanded in terms of monochromatic modes. In this case the number states attributed to a mode of frequency ω may be called photon-number states since – for a mode of frequency ω – they are states with a precise number n of energy quanta $\hbar\omega$ which are commonly denoted as photons. The average energy of the mode in a number state is obtained by taking the expectation value $\langle n|\hat{H}|n\rangle$ with $\hat{H}$ according to Eq. (3.2). Since $\langle n|\hat{n}|n\rangle = n$, we readily obtain

$$\langle n|\hat{H}|n\rangle = \hbar\omega\left(n + \tfrac{1}{2}\right), \tag{3.23}$$

which shows that an energy $\hbar\omega$ is indeed associated with each photon. However, in the absence of photons, $n=0$, i. e., for the ground state, there is still the zero-point energy of the associated harmonic oscillator left: $\langle 0|\hat{H}|0\rangle = \hbar\omega/2$. This is due to the fact that quantities such as the coordinate and the momentum of the oscillator still reveal fluctuations in the ground state.

A rough measure of the fluctuation of a quantity $\hat{O}$ is the variance $\langle(\Delta\hat{O})^2\rangle$, where $\Delta\hat{O} = \hat{O} - \langle\hat{O}\rangle$. Since the number of quanta is well defined in a number state, the number variance for such a state of course vanishes,

$$\langle n|(\Delta\hat{n})^2|n\rangle = \langle n|\hat{n}^2|n\rangle - \langle n|\hat{n}|n\rangle^2 = 0, \tag{3.24}$$

as also does the energy variance.

Let us now consider the expectation value and the variance of a coordinate-like quantity such as the electric field of a single-mode radiation field,

$$\hat{\mathbf{E}}(\mathbf{r}) = i\omega[\hat{a}\,\mathbf{A}(\mathbf{r}) - \hat{a}^\dagger\mathbf{A}^*(\mathbf{r})] \tag{3.25}$$

[cf. Eq. (2.70)], where $\mathbf{A}(\mathbf{r})$ is the corresponding mode function. Using Eqs (3.19)–(3.21), we readily prove that when the mode is prepared in a photon-number state, then the mean value of the electric-field strength vanishes,

$$\langle n|\hat{\mathbf{E}}(\mathbf{r})|n\rangle = 0. \tag{3.26}$$

That is, a photon-number state is far from representing a (nonfluctuating) classical wave which would naturally reveal a nonvanishing electric field. In view of particle–wave dualism, this type of state is closely related to the particle nature of the radiation rather than to its wave nature. It will be seen later (Chapter 8) that it is a nontrivial problem to experimentally prepare radiation field modes in photon-number states. The expectation value of the intensity

$$\hat{I}(\mathbf{r}) = \hat{\mathbf{E}}^{(-)}(\mathbf{r})\hat{\mathbf{E}}^{(+)}(\mathbf{r}) \tag{3.27}$$

in a photon-number state is proportional to the number of photons in the field:

$$\langle n|\hat{I}(\mathbf{r})|n\rangle = \omega^2|A(\mathbf{r})|^2\langle n|\hat{a}^\dagger\hat{a}|n\rangle = \omega^2|A(\mathbf{r})|^2 n. \tag{3.28}$$

Accordingly, the variance of the kth component of the electric-field strength is found to be

$$\langle n|[\Delta\hat{E}_k(\mathbf{r})]^2|n\rangle = \omega^2|A_k(\mathbf{r})|^2(2n+1), \tag{3.29}$$

from which the fluctuation of the electric field is seen to increase with the photon number n. The minimum noise is obtained for $n=0$, that is, in the case of the vacuum field we have

$$\langle 0|[\Delta\hat{E}_k(\mathbf{r})]^2|0\rangle = \omega^2|A_k(\mathbf{r})|^2. \tag{3.30}$$

This clearly shows that even in the vacuum case, when no photons are present, there are quantum fluctuations of the field.

3.1.2 Multi-mode number states

We now turn to the more general case of a multi-mode system and remember that the Hamiltonian is additively composed of independent single-mode Hamiltonians [Eq. (3.1) together with Eq. (3.2)]. The additivity of the Hamiltonian leads to the fact that its eigenstates are simply products of the single-mode eigenstates, that is, $\hat{H}|\{n_\lambda\}\rangle = E_{\{n_\lambda\}}|\{n_\lambda\}\rangle$, where the eigenstates and energies are given by

$$|\{n_\lambda\}\rangle = \prod_\lambda |n_\lambda\rangle, \tag{3.31}$$

$$E_{\{n_\lambda\}} = \sum_\lambda \hbar\omega_\lambda\left(n_\lambda + \tfrac{1}{2}\right). \tag{3.32}$$

In the case of a radiation field, for example, one may consider an experiment where the total number of photons is measured regardless of their frequency or polarization, i. e., regardless of which mode they are in. The measured operator would then be the total-number operator, defined by

$$\hat{N} = \sum_\lambda \hat{n}_\lambda. \tag{3.33}$$

We can easily see that the total-number operator $\hat{N}$ commutes with the Hamiltonian (3.1) [together with Eq. (3.2)] and therefore has the same eigenvectors $|\{n_\lambda\}\rangle$,

$$\hat{N}|\{n_\lambda\}\rangle = N|\{n_\lambda\}\rangle, \tag{3.34}$$

with the total number of photons in the state $|\{n_\lambda\}\rangle$ being

$$N = \sum_\lambda n_\lambda. \tag{3.35}$$

However, since different combinations of numbers n_λ can lead to the same total number N, there is a degeneracy with respect to the operator $\hat{N}$. Taking into account Λ different modes, it can be readily seen that there are Λ^N possibilities of occupying the different single modes to obtain the total number N.

The noise properties of a multi-mode system in a number state are quite similar to those of a single-mode system. In particular, the variance of the total-photon number of a multi-mode radiation field vanishes, as does the mean value of the electric field:

$$\langle\{n_\lambda\}|(\Delta\hat{N})^2|\{n_\lambda\}\rangle = 0, \tag{3.36}$$

$$\langle\{n_\lambda\}|\hat{\mathbf{E}}(\mathbf{r})|\{n_\lambda\}\rangle = 0, \tag{3.37}$$

where $\hat{\mathbf{E}}(\mathbf{r})$ is now meant to be the multi-mode electric-field strength according to Eq. (2.70).

3.2
Coherent states

The quantum states that come closest to the classical ideal are the coherent states. As their name suggests, they indeed show a coherent amplitude, i. e., they reveal the classically expected oscillatory behavior of the harmonic-oscillator coordinates. In order to derive the coherent states and to provide the tools for the squeezed states (Section 3.3), we follow an approach based on unitary transformations. For the sake of clarity we again start from a single-mode system. Performing a unitary transformation $\hat{U}$ ($\hat{U}^\dagger{=}\hat{U}^{-1}$), the operator $\hat{a}$ transforms to $\hat{a}'$ as

$$\hat{a}' = \hat{U}\hat{a}\hat{U}^\dagger, \tag{3.38}$$

whereas the transformed number states $|n\rangle'$ read

$$|n\rangle' = \hat{U}|n\rangle. \tag{3.39}$$

Clearly, the transformed operators $\hat{a}'$ and $\hat{a}'^{\dagger}$ again obey the bosonic commutator relations (3.4) and (3.5). Defining the transformed number operator $\hat{n}' = \hat{a}'^{\dagger}\hat{a}'$ and taking into account Eqs (3.6), (3.19) and (3.20) we readily find the eigenvalue equation for the transformed number operator,

$$\hat{n}'|n\rangle' = n|n\rangle', \tag{3.40}$$

and the actions of the transformed creation and annihilation operators,

$$\hat{a}'^{\dagger}|n\rangle' = \sqrt{n+1}\,|n+1\rangle', \tag{3.41}$$

$$\hat{a}'|n\rangle' = \sqrt{n}\,|n-1\rangle'. \tag{3.42}$$

The transformed operators and states thus reveal the same algebraic relations as the original ones and therefore can also be used as a complete set of states to span the Hilbert space. In particular, from Eq. (3.42) we see that

$$\hat{a}'|0\rangle' = 0, \tag{3.43}$$

which defines a new ground state with respect to the transformed operators. Clearly, although their algebraic relations do not change, the physical properties of the transformed number states $|n\rangle'$ may drastically differ from the original number states $|n\rangle$. This, however, depends solely on the actual form of the applied unitary transformation $\hat{U}$.

The transformation leading to the coherent states is implemented by the displacement operator $\hat{D}(\alpha)$,

$$\hat{U} \equiv \hat{D}(\alpha) = \exp(\alpha\hat{a}^{\dagger} - \alpha^{*}\hat{a}), \tag{3.44}$$

with α being a complex c-number variable. Using Eq. (C.27), we may factorize the displacement operator to obtain its normally and anti-normally ordered forms, respectively,

$$\hat{D}(\alpha) = e^{\alpha\hat{a}^{\dagger}}e^{-\alpha^{*}\hat{a}}e^{-|\alpha|^{2}/2}, \tag{3.45}$$

$$\hat{D}(\alpha) = e^{-\alpha^{*}\hat{a}}e^{\alpha\hat{a}^{\dagger}}e^{|\alpha|^{2}/2}. \tag{3.46}$$

We may therefore write the transformed annihilation operator $\hat{a}'$ as

$$\hat{a}' = \hat{D}(\alpha)\hat{a}\hat{D}^{\dagger}(\alpha) = e^{\alpha\hat{a}^{\dagger}}e^{-\alpha^{*}\hat{a}}\hat{a}\,e^{\alpha^{*}\hat{a}}e^{-\alpha\hat{a}^{\dagger}} = e^{\alpha\hat{a}^{\dagger}}\hat{a}\,e^{-\alpha\hat{a}^{\dagger}}, \tag{3.47}$$

from which, together with the relation (C.9), it follows that

$$\hat{a}' = \hat{a} - \alpha. \tag{3.48}$$

This result now enables us to rewrite the definition of the transformed ground state (3.43) as

$$(\hat{a} - \alpha)\hat{D}(\alpha)|0\rangle = 0. \tag{3.49}$$

Denoting by $|\alpha\rangle = |0\rangle'$ the transformed ground state which depends parametrically on α,

$$|\alpha\rangle = \hat{D}(\alpha)|0\rangle, \tag{3.50}$$

we see from Eq. (3.49) that for each complex number α the state $|\alpha\rangle$ is a right-hand eigenstate of the non-Hermitian annihilation operator $\hat{a}$ with eigenvalue α,

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle. \tag{3.51}$$

From Eq. (3.51) we further see that correspondingly $\langle\alpha|$ is a left-hand eigenstate of $\hat{a}^\dagger$,

$$\langle\alpha|\hat{a}^\dagger = \langle\alpha|\alpha^*. \tag{3.52}$$

The states $|\alpha\rangle$, being normalized to unity ($\langle\alpha|\alpha\rangle = 1$), are called coherent states or Glauber states [Schrödinger (1926); Klauder (1960); Glauber (1963a,b,c)]. The amplitude α determines a point in a complex phase space which corresponds to a coherent amplitude of the corresponding harmonic oscillation, i. e., $\langle\alpha|\hat{a}|\alpha\rangle = \alpha$. This phase-space amplitude or coherent excitation can be changed by use of the displacement operator, which can be seen by first considering the action of two subsequent displacements,

$$\hat{D}(\alpha)\hat{D}(\beta) = \hat{D}(\alpha+\beta)\exp[i\mathrm{Im}(\alpha\beta^*)]\,. \tag{3.53}$$

From this equation together with Eq. (3.50) we see that, apart from a phase factor, the action of a displacement operator $\hat{D}(\beta)$ on a coherent state $|\alpha\rangle$ creates a new coherent state with amplitude $\alpha+\beta$,

$$\hat{D}(\beta)|\alpha\rangle = e^{-i\,\mathrm{Im}(\alpha\beta^*)}|\alpha+\beta\rangle, \tag{3.54}$$

i. e., the operator $\hat{D}(\beta)$ displaces the phase-space amplitude of the coherent state by the amount β. In particular, this also shows that the ground state $|0\rangle$ can be regarded as being the coherent state of amplitude $\alpha = 0$, from which, by application of the displacement operator (3.44), all possible coherent states can be obtained, in agreement with Eq. (3.50).

The action of $\hat{a}^\dagger$ on $|\alpha\rangle$ and $\hat{a}$ on $\langle\alpha|$, respectively, can be derived as follows. Applying $\hat{a}^\dagger$ to $|\alpha\rangle = \hat{D}(\alpha)|0\rangle$ and using the normally ordered form of the displacement operator (3.45) yields

$$\begin{aligned}\hat{a}^\dagger|\alpha\rangle &= \hat{a}^\dagger \exp\big[\big(\hat{a}^\dagger - \tfrac{1}{2}\,\alpha^*\big)\alpha\big]\exp(-\alpha^*\hat{a})\,|0\rangle = \hat{a}^\dagger \exp\big[\big(\hat{a}^\dagger - \tfrac{1}{2}\,\alpha^*\big)\alpha\big]|0\rangle \\ &= \left(\frac{\partial}{\partial\alpha} + \frac{\alpha^*}{2}\right)\exp\big[\big(\hat{a}^\dagger - \tfrac{1}{2}\,\alpha^*\big)\alpha\big]|0\rangle,\end{aligned} \tag{3.55}$$

and hence

$$\hat{a}^{\dagger}|\alpha\rangle = \left(\frac{\partial}{\partial\alpha} + \frac{\alpha^*}{2}\right)|\alpha\rangle. \tag{3.56}$$

Accordingly, applying $\hat{a}$ to $\langle\alpha|$ and using the anti-normally ordered form (3.46) yields

$$\langle\alpha|\hat{a} = \langle\alpha|\left(\frac{\overleftarrow{\partial}}{\partial\alpha^*} + \frac{\alpha}{2}\right), \tag{3.57}$$

where the derivative is supposed to act to the left side.

Clearly, the coherent states $|\alpha\rangle$ can be expanded in terms of the number states $|n\rangle$ by use of their completeness relation (3.22),

$$|\alpha\rangle = \sum_{n=0}^{\infty} |n\rangle\langle n|\alpha\rangle. \tag{3.58}$$

The expansion coefficients $\langle n|\alpha\rangle$ can be calculated by means of Eqs (3.18) and (3.50), resulting in

$$\langle n|\alpha\rangle = \frac{\alpha^n}{\sqrt{n!}}\, e^{-|\alpha|^2/2}. \tag{3.59}$$

From Eqs (3.58) and (3.59) we immediately find that the number distribution of a coherent state is a Poissonian:

$$|\langle n|\alpha\rangle|^2 = \frac{(|\alpha|^2)^n}{n!}\, e^{-|\alpha|^2}, \tag{3.60}$$

with mean value and variance both being given by $|\alpha|^2$,

$$\langle\alpha|\hat{n}|\alpha\rangle = \langle\alpha|(\Delta\hat{n})^2|\alpha\rangle = |\alpha|^2. \tag{3.61}$$

Hence the number of quanta is a fluctuating quantity for a coherent state $|\alpha\rangle$.

We recall that the coherent states are eigenstates of a non-Hermitian operator. In comparison with the eigenstates of Hermitian operators, they therefore exhibit some unusual features. They are over-complete and nonorthogonal. Let us consider two coherent states $|\alpha\rangle$ and $|\beta\rangle$ (with $\alpha \neq \beta$) and calculate their overlap $\langle\beta|\alpha\rangle$. From Eqs (3.51) and (3.57) we obtain the relation

$$\langle\beta|\hat{a}|\alpha\rangle = \alpha\langle\beta|\alpha\rangle = \left(\frac{\partial}{\partial\beta^*} + \frac{\beta}{2}\right)\langle\beta|\alpha\rangle, \tag{3.62}$$

which represents a differential equation for $\langle\beta|\alpha\rangle$:

$$\frac{\partial}{\partial\beta^*}\langle\beta|\alpha\rangle = \left(\alpha - \frac{\beta}{2}\right)\langle\beta|\alpha\rangle. \tag{3.63}$$

Taking into account the boundary condition $\langle\alpha|\alpha\rangle = 1$, the solution is obtained as

$$\langle\beta|\alpha\rangle = \exp\left[-\tfrac{1}{2}|\alpha-\beta|^2 + \tfrac{1}{2}(\alpha\,\beta^* - \alpha^*\beta)\right], \tag{3.64}$$

and hence the squared modulus reads

$$|\langle\beta|\alpha\rangle|^2 = \exp(-|\alpha-\beta|^2). \tag{3.65}$$

Equation (3.65) clearly shows that for $\alpha \neq \beta$ the states $|\alpha\rangle$ and $|\beta\rangle$ are indeed not orthogonal to each other. However, if the values of α and β are sufficiently separated, so that $|\alpha-\beta| \gg 1$, they may be regarded as being approximately orthogonal.

To show that the coherent states resolve the identity, we recall the completeness relation for the number states, which enables us to write

$$\int \mathrm{d}^2\alpha\,|\alpha\rangle\langle\alpha| = \sum_{n,m=0}^{\infty} \int \mathrm{d}^2\alpha\,|n\rangle\langle n|\alpha\rangle\langle\alpha|m\rangle\langle m|, \tag{3.66}$$

where the integration is performed over the real and imaginary parts, $\alpha' \equiv \mathrm{Re}\,\alpha$ and $\alpha'' \equiv \mathrm{Im}\,\alpha$, respectively, of $\alpha = \alpha' + i\alpha''$,

$$\mathrm{d}^2\alpha = \mathrm{d}\alpha'\mathrm{d}\alpha''. \tag{3.67}$$

We now use Eq. (3.59) and rewrite Eq. (3.66) as

$$\int \mathrm{d}^2\alpha\,|\alpha\rangle\langle\alpha| = \sum_{n,m=0}^{\infty} \frac{|m\rangle\langle n|}{\sqrt{m!\,n!}} \int \mathrm{d}^2\alpha\,\alpha^m\alpha^{*n}e^{-|\alpha|^2}. \tag{3.68}$$

The integral in Eq. (3.68) can be evaluated to be

$$\int \mathrm{d}^2\alpha\,\alpha^m\alpha^{*n}e^{-|\alpha|^2} = \pi n!\delta_{mn}, \tag{3.69}$$

and we arrive at

$$\int \mathrm{d}^2\alpha\,|\alpha\rangle\langle\alpha| = \pi\sum_{n=0}^{\infty}|n\rangle\langle n| = \pi\hat{I}, \tag{3.70}$$

from which we see that the identity can be resolved as

$$\frac{1}{\pi}\int \mathrm{d}^2\alpha\,|\alpha\rangle\langle\alpha| = \hat{I}. \tag{3.71}$$

That is to say, any state $|\psi\rangle$ can be expanded in terms of the coherent states as follows:

$$|\psi\rangle = \frac{1}{\pi}\int \mathrm{d}^2\alpha\,|\alpha\rangle\langle\alpha|\psi\rangle. \tag{3.72}$$

From Eq. (3.72) together with Eq. (3.65) we see that there is a nontrivial expansion of a coherent state in terms of coherent states, which indicates the over-completeness of the coherent states. To demonstrate the over-completeness more explicitly, let us consider a subset of coherent states whose modulus r of the complex amplitude, $\alpha = re^{i\varphi}$, is chosen to be constant. This represents the set of coherent states on a circle in phase space. Using Eq. (3.59) the number representation of these states is given as

$$|re^{i\varphi}\rangle = e^{-r^2/2} \sum_{n=0}^{\infty} \frac{r^n}{\sqrt{n!}} e^{in\varphi} |n\rangle. \tag{3.73}$$

Fourier transforming this equation with respect to the phase φ, we readily derive a representation of the number states $|n\rangle$ $(n = 0, 1, \ldots)$ in terms of the coherent states on a circle,

$$|n\rangle = \frac{\sqrt{n!}}{2\pi\, r^n} e^{r^2/2} \int_0^{2\pi} \mathrm{d}\varphi\, e^{-in\varphi} |re^{i\varphi}\rangle. \tag{3.74}$$

This result reveals that the complete number-state basis can be expressed in terms of the coherent states on any chosen circle in phase space. Equivalently, complete sets of coherent states can also be chosen on other contours, for example on a straight line [Adam, Földesi, and Janszky (1994)].

3.2.1
Statistics of the coherent states

To illustrate the main difference between coherent states and number states, let us focus on a radiation-field mode. We have already seen that in the case of the mode being in a coherent state $|\alpha\rangle$ the probability of finding n photons obeys a Poissonian distribution [Eq. (3.60)]. Hence both the mean number of photons and its variance are given by the squared modulus of the complex amplitude, $|\alpha|^2$. Next let us again consider a quantity of the type of the electric-field strength. From Eq. (3.25) together with Eqs (3.51) and (3.52) we derive for the mean value of the kth component of the electric-field strength

$$\langle\alpha|\hat{E}_k(\mathbf{r})|\alpha\rangle = i\omega[A_k(\mathbf{r})\alpha - A_k^*(\mathbf{r})\alpha^*]. \tag{3.75}$$

That is, when the mode is prepared in a coherent state, then the mean electric-field strength looks like the electric-field strength of a coherent, classical mode with (complex) amplitude α. Notwithstanding this resemblance, there is a fundamental difference between a classical and a quantum mode in a coherent state, because of the vacuum noise inherent in the quantum system. Calculating the variance of the kth component of the electric-field strength, we easily derive

$$\langle\alpha|[\Delta\hat{E}_k(\mathbf{r})]^2|\alpha\rangle = \omega^2|A_k(\mathbf{r})|^2. \tag{3.76}$$

Comparing this with Eq. (3.30), we see that the noise of the electric field is indeed determined by the vacuum level, independent of the field amplitude. Rewriting Eq. (3.75) as

$$\langle\alpha|\hat{E}_k(\mathbf{r})|\alpha\rangle = 2\omega|A_k(\mathbf{r})||\alpha| \sin\varphi_{E_k}, \tag{3.77}$$

where φ_{E_k} is the phase of the (kth component of the) mean value of the electric-field strength, and using Eq. (3.76), we can easily calculate the relative noise of the electric field, obtaining

$$\left\{\frac{\langle\alpha|[\Delta\hat{E}_k(\mathbf{r})]^2|\alpha\rangle}{[\langle\alpha|\hat{E}_k(\mathbf{r})|\alpha\rangle]^2}\right\}^{\frac{1}{2}} = \frac{1}{2|\alpha||\sin\varphi_{E_k}|}\,. \tag{3.78}$$

Equation (3.78) reveals that (for $\sin\varphi_{E_k} \neq 0$) the relative noise decreases with increasing absolute value of α or, according to Eq. (3.61), with the square root of the mean photon number. The coherent states $|\alpha\rangle$ may therefore be regarded as being those quantum states that correspond most closely to classical, coherent waves. Without going into the detail of quantum coherence theory [see, e. g., Peřina (1985); Mandel and Wolf (1995)], we note that with a radiation field being prepared in a coherent state, normally ordered correlation functions factorize perfectly, that is, the coherence condition is satisfied up to any order:

$$\langle\alpha|\big(\hat{E}_k^{(-)}\big)^m\big(\hat{E}_k^{(+)}\big)^n|\alpha\rangle = \big(\langle\alpha|\hat{E}_k^{(-)}|\alpha\rangle\big)^m\big(\langle\alpha|\hat{E}_k^{(+)}|\alpha\rangle\big)^n. \tag{3.79}$$

3.2.2 Multi-mode coherent states

The extension of the concept of coherent states to multi-mode systems is straightforward. Similar to the case of number states, the multi-mode coherent states $|\{\alpha_\lambda\}\rangle$ are simply obtained by taking the (direct) product of single-mode coherent states, that is,

$$|\{\alpha_\lambda\}\rangle = \prod_\lambda |\alpha_\lambda\rangle, \tag{3.80}$$

and the identity operator in the multi-mode Hilbert space then reads

$$\prod_\lambda \left(\frac{1}{\pi}\int \mathrm{d}^2\alpha_\lambda\, |\alpha_\lambda\rangle\langle\alpha_\lambda|\right) = \hat{I}. \tag{3.81}$$

In view of a multi-mode radiation field, the mean value of the total number of photons in all modes, $\hat{N} = \sum_\lambda \hat{n}_\lambda$, and the corresponding photon-number variance are given by

$$\langle\{\alpha_\lambda\}|\hat{N}|\{\alpha_\lambda\}\rangle = \sum_\lambda |\alpha_\lambda|^2, \tag{3.82}$$

$$\langle\{\alpha_\lambda\}|(\Delta\hat{N})^2|\{\alpha_\lambda\}\rangle = \langle\{\alpha_\lambda\}|\hat{N}|\{\alpha_\lambda\}\rangle, \tag{3.83}$$

and the mean value and the variance of the kth component of the electric-field strength are, respectively,

$$\langle\{\alpha_\lambda\}|\hat{E}_k(\mathbf{r})|\{\alpha_\lambda\}\rangle = \sum_\lambda i\omega_\lambda\left[A_{\lambda,k}(\mathbf{r})\alpha_\lambda - A^*_{\lambda,k}(\mathbf{r})\alpha^*_\lambda\right], \tag{3.84}$$

$$\langle\{\alpha_\lambda\}|[\Delta\hat{E}_k(\mathbf{r})]^2|\{\alpha_\lambda\}\rangle = \sum_\lambda \omega^2_\lambda|A_{\lambda,k}(\mathbf{r})|^2. \tag{3.85}$$

To illustrate how a classical light pulse emerges from a radiation field in a coherent state $|\{\alpha_\lambda\}\rangle$, let us consider a multi-mode radiation field propagating in the positive x direction in free space. To take into account the temporal evolution of the (free) radiation field, we recall that in the Heisenberg picture the photon annihilation evolves as

$$\hat{a}_\lambda(t) = \hat{a}_\lambda e^{-i\omega_\lambda t}, \tag{3.86}$$

where $\hat{a}_\lambda = \hat{a}_\lambda(0)$ is the annihilation operator at some initial time $t=0$. Assuming that the (nonevolving) initial state vector is a multi-mode coherent state as given in Eq. (3.80), and using the traveling-wave mode functions of frequency ω_l, polarization $\mathbf{e}_{l,\sigma}$ and quantization volume $\mathcal{AL}$,

$$\mathbf{A}_\lambda(\mathbf{r}) \;\mapsto\; \mathbf{A}_{l\sigma}(x) = \left(\frac{\hbar}{2\epsilon_0\omega_l\mathcal{AL}}\right)^{\frac{1}{2}} \mathbf{e}_{l,\sigma}e^{i\omega_l x/c}, \tag{3.87}$$

we derive from Eqs (3.86) and (3.87) the kth component of the electric-field operator as

$$\hat{E}_k(x,t) = i\sum_{l,\sigma}\sqrt{\frac{\hbar\omega_l}{2\epsilon_0\mathcal{AL}}}\,(\mathbf{e}_{l,\sigma})_k\hat{a}_{l,\sigma}e^{-i\omega_l(t-x/c)} + \text{H.c.} \tag{3.88}$$

[cf. Eqs (2.87) and (2.88)]. We now perform the limit of infinite propagation length, $\mathcal{L}\to\infty$, while the diameter or beam waist $\mathcal{A}$ is held constant. With increasing $\mathcal{L}$ the modes become more and more dense in the frequency domain, because $\omega_l = 2\pi cl/\mathcal{L}$. Defining the operators

$$\hat{a}_\sigma(\omega) = \lim_{\mathcal{L}\to\infty}\frac{\hat{a}_{l,\sigma}}{\sqrt{\Delta\omega}} \tag{3.89}$$

with $\Delta\omega = 2\pi c/\mathcal{L}$ [cf. Eqs (2.89) and (2.90)], we see that in the limit $\mathcal{L}\to\infty$ the l sum in Eq. (3.88) can be written as an integral and the positive-frequency part of the electric-field operator becomes

$$\hat{E}^{(+)}_k(x,t) = i\sum_\sigma\int_0^\infty \mathrm{d}\omega\,\sqrt{\frac{\hbar\omega}{4\pi\epsilon_0 c\mathcal{A}}}\,[\mathbf{e}_\sigma(\omega)]_k e^{-i\omega(t-x/c)}\hat{a}_\sigma(\omega). \tag{3.90}$$

Correspondingly, the mean electric field in a coherent state reads

$$\langle \hat{E}_k(x,t)\rangle_{\mathrm{coh}} = i\sum_{\sigma}\int_0^{\infty} \mathrm{d}\omega \sqrt{\frac{\hbar\omega}{4\pi\epsilon_0 c\mathcal{A}}}\,[\mathbf{e}_\sigma(\omega)]_k \alpha_\sigma(\omega) e^{-i\omega(t-x/c)} + \text{c.c.}. \tag{3.91}$$

Equation (3.91) is capable of describing (the kth component of the electric-field strength of) a coherent light pulse. To give an example, let us consider the case where the polarization unit vectors $\mathbf{e}_\sigma(\omega)$ are independent of frequency and suppose that the mode amplitudes are polarization independent,

$$\alpha_\sigma(\omega) \equiv \alpha(\omega) = |\alpha(\omega)|e^{i\overline{\varphi}}, \tag{3.92}$$

with the photon spectrum being of Gaussian form with center frequency $\overline{\omega}$ and spectral width $\Delta\omega \ll \overline{\omega}$,

$$|\alpha(\omega)|^2 = \frac{\langle \hat{N}\rangle_{\mathrm{coh}}}{\sqrt{2\pi}\,\Delta\omega} \exp\left[-\frac{1}{2}\left(\frac{\omega-\overline{\omega}}{\Delta\omega}\right)^2\right], \tag{3.93}$$

where $\langle \hat{N}\rangle_{\mathrm{coh}}$ is the total number of photons of the light pulse:

$$\langle \hat{N}\rangle_{\mathrm{coh}} = \int_0^{\infty} \mathrm{d}\omega\, |\alpha(\omega)|^2. \tag{3.94}$$

Combining Eqs (3.91)–(3.93) and taking into account that the spectral width of the electric-field strength is small compared with the center frequency ($\Delta\omega \ll \omega$), we obtain for the mean electric field

$$\langle \hat{E}_k(x,t)\rangle_{\mathrm{coh}} = 2\sqrt{\frac{\Delta\omega\,\hbar\overline{\omega}\langle \hat{N}\rangle_{\mathrm{coh}}}{\epsilon_0 c\mathcal{A}\sqrt{2\pi}}} \exp\left\{-\left[\Delta\omega\left(t-\frac{x}{c}\right)\right]^2\right\} \sin\left[\overline{\omega}\left(t-\frac{x}{c}\right)-\overline{\varphi}\right], \tag{3.95}$$

which represents an unpolarized, coherent Gaussian light pulse traveling in positive x direction.

3.2.3
Displaced number states

At this point it should be noted that the coherent states are a special class of states with respect to the transformation given in Eqs (3.39) and (3.44), since they are defined by the action of the displacement operator on the ground state $|0\rangle$. A broader class of states is obtained by considering the transformed states that emerge from the application of the displacement operator on arbitrary number states $|n\rangle$. Such states are denoted as displaced number states and they are defined via Eqs (3.39) and (3.44) as

$$|n,\alpha\rangle = \hat{D}(\alpha)|n\rangle. \tag{3.96}$$

From the general transformed eigenvalue equation (3.40) we can see that the displaced number states are eigenstates of the displaced number operator, i. e., $\hat{n}(\alpha)|n,\alpha\rangle = n|n,\alpha\rangle$, with

$$\hat{n}(\alpha) = \hat{D}(\alpha)\hat{n}\hat{D}^{\dagger}(\alpha) = (\hat{a}^{\dagger} - \alpha^*)(\hat{a} - \alpha), \tag{3.97}$$

where we have made use of Eq. (3.48).

Clearly, for a fixed displacement α these states are orthonormal, just as the number states are:

$$\langle n,\alpha|m,\alpha\rangle = \langle n|\hat{D}^{\dagger}(\alpha)\hat{D}(\alpha)|m\rangle = \delta_{nm}, \tag{3.98}$$

and for arbitrary α the identity operator can be resolved as

$$\sum_{n=0}^{\infty} |n,\alpha\rangle\langle n,\alpha| = \hat{I}. \tag{3.99}$$

Moreover, for arbitrary n the identity can also be resolved by an integral over the displacement amplitude, in analogy with Eq. (3.71),

$$\frac{1}{\pi}\int \mathrm{d}^2\alpha\, |n,\alpha\rangle\langle n,\alpha| = \hat{I}. \tag{3.100}$$

Finally, their scalar product with number states can be shown to be expressible in terms of the Laguerre polynomials $\mathrm{L}_n^{(k)}(x)$ as

$$\langle n|m,\alpha\rangle = (-\alpha^*)^{m-n}\sqrt{\frac{n!}{m!}}\,\mathrm{L}_n^{(m-n)}\big(|\alpha|^2\big)\, e^{-|\alpha|^2/2} \qquad (m \geq n), \tag{3.101}$$

and $\langle n|m,\alpha\rangle = (\langle m|n,-\alpha\rangle)^*$ for the coefficients with $m < n$.

3.3
Squeezed states

Another important class of quantum states are the squeezed states or more precisely the quadrature-squeezed states. For the purpose of deriving these states we return to the unitary transformation, Eqs (3.38) and (3.39), which was employed to derive the coherent states, and assume that the unitary operator $\hat{U}$ is now the squeeze operator,

$$\hat{U} \equiv \hat{S}(\xi) = \exp\big[\tfrac{1}{2}(\xi^*\hat{a}^2 - \xi\hat{a}^{\dagger 2})\big], \tag{3.102}$$

with ξ – the squeezing parameter – being a complex number. To obtain the relation between the transformed annihilation and creation operators $\hat{a}'$, $\hat{a}'^{\dagger}$ and the original ones, it is convenient to define the operators

$$\hat{G}(z) = \hat{S}^{z}(\xi), \tag{3.103}$$

$$\hat{a}(z) = \hat{G}(z)\hat{a}\hat{G}^{\dagger}(z), \tag{3.104}$$

where z is a real number (the parameter ξ has been omitted for notational convenience). Comparing Eqs (3.38) and (3.104) we see that the original and the transformed annihilation operators are recovered from $\hat{a}(z)$ for $z=0$ and $z=1$, respectively,

$$\hat{a}(z)|_{z=0} = \hat{a}, \tag{3.105}$$
$$\hat{a}(z)|_{z=1} = \hat{a}'. \tag{3.106}$$

Using Eqs (3.102) and (3.103) the derivative of the operator (3.104) with respect to z can be obtained as

$$\begin{aligned}\frac{\mathrm{d}\hat{a}(z)}{\mathrm{d}z} &= \frac{\mathrm{d}\hat{G}(z)}{\mathrm{d}z}\,\hat{a}\hat{G}^\dagger(z) + \hat{G}(z)\hat{a}\,\frac{\mathrm{d}\hat{G}^\dagger(z)}{\mathrm{d}z} \\ &= \big[\hat{a}(z), \tfrac{1}{2}\{\xi\hat{a}^{\dagger 2}(z) - \xi^*\hat{a}^2(z)\}\big] = \tfrac{1}{2}\xi\big[\hat{a}(z), \hat{a}^{\dagger 2}(z)\big].\end{aligned} \tag{3.107}$$

Since Eq. (3.104) describes a unitary transformation, the operators $\hat{a}(z)$ and $\hat{a}^\dagger(z)$ again obey the bosonic commutator relation (3.4) and we obtain from Eq. (3.107)

$$\frac{\mathrm{d}\hat{a}(z)}{\mathrm{d}z} = \xi\hat{a}^\dagger(z), \quad \frac{\mathrm{d}\hat{a}^\dagger(z)}{\mathrm{d}z} = \xi^*\hat{a}(z). \tag{3.108}$$

The solution to the differential equations (3.108) reads

$$\hat{a}(z) = \hat{c}_1 e^{|\xi|z} + \hat{c}_2 e^{-|\xi|z}, \tag{3.109}$$

where the operators $\hat{c}_1$ and $\hat{c}_2$ are determined, according to Eqs (3.105) and (3.108), by the initial conditions

$$\hat{a}(z)|_{z=0} = \hat{c}_1 + \hat{c}_2 = \hat{a}, \tag{3.110}$$
$$\left.\frac{\mathrm{d}\hat{a}(z)}{\mathrm{d}z}\right|_{z=0} = |\xi|(\hat{c}_1 - \hat{c}_2) = \xi\hat{a}^\dagger. \tag{3.111}$$

Combining Eqs (3.109)–(3.111), after some algebra we finally arrive at ($\varphi_\xi = \arg(\xi)$)

$$\hat{a}(z) = \hat{a}\cosh(|\xi|z) + \hat{a}^\dagger e^{i\varphi_\xi}\sinh(|\xi|z), \tag{3.112}$$

from which we obtain, according to Eq. (3.106), the transformed operators $\hat{a}'$ and $\hat{a}'^\dagger$ in terms of the original ones:[1]

$$\hat{a}' = \mu\hat{a} + \nu\hat{a}^\dagger, \tag{3.113}$$
$$\hat{a}'^\dagger = \mu\hat{a}^\dagger + \nu^*\hat{a}, \tag{3.114}$$

1) Note that this is a SU(1,1) group transformation.

where the parameters μ and ν are defined by

$$\mu = \cosh|\xi|, \tag{3.115}$$

$$\nu = e^{i\varphi_\xi}\sinh|\xi|. \tag{3.116}$$

As mentioned before, since the operators $\hat{a}'$ and $\hat{a}'^\dagger$ are obtained from the operators $\hat{a}$ and $\hat{a}^\dagger$ by a unitary transformation, the (equal-time) commutator relation is preserved,

$$[\hat{a}', \hat{a}'^\dagger] = 1. \tag{3.117}$$

Inserting Eqs (3.113) and (3.114) into Eq. (3.117), it follows that the parameters μ and ν obey the following constraint:

$$\mu^2 - |\nu|^2 = 1, \tag{3.118}$$

which apparently is provided by their definitions (3.115) and (3.116).

As a unitary transformation $\hat{U}$ that can be used to define the squeezed coherent states, in a similar way to the coherent states (cf. Section 3.2), let us consider now the unitary transformation

$$\hat{U}(\xi, \beta) = \hat{S}(\xi)\hat{D}(\beta). \tag{3.119}$$

It first coherently displaces by an amplitude β and then squeezes with squeezing parameter ξ. Analogously to the definition of the coherent states [Eq. (3.50)], we may now define the states $|\xi, \beta\rangle$ by applying the transformation (3.119) onto the ground state $|0\rangle$,

$$|\xi, \beta\rangle = \hat{U}(\xi, \beta)|0\rangle = \hat{S}(\xi)\hat{D}(\beta)|0\rangle = \hat{S}(\xi)|\beta\rangle. \tag{3.120}$$

The states $|\xi, \beta\rangle$ (sometimes denoted by $|\mu, \nu; \beta\rangle$) are called squeezed coherent states [Stoler (1970, 1971); Yuen (1976); for reviews see Walls (1983); Loudon and Knight (1987)].

From the relations (3.120) we see that applying the displacement operator $\hat{D}(\beta)$ to the vacuum state $|0\rangle$ and then applying the squeeze operator $\hat{S}(\xi)$ to the resulting coherent state $|\beta\rangle$ yields the squeezed coherent state $|\xi, \beta\rangle$. However, we may arrive at the same result if we first apply the squeeze operator $\hat{S}(\xi)$ to the vacuum state $|0\rangle$ in order to generate the squeezed ground (or vacuum) state,

$$|\xi, 0\rangle = \hat{S}(\xi)|0\rangle \tag{3.121}$$

and then apply the transformed displacement operator $\hat{D}'(\beta)$ to this state:

$$|\xi, \beta\rangle = \hat{D}'(\beta)|\xi, 0\rangle = \hat{D}'(\beta)\hat{S}(\xi)|0\rangle, \tag{3.122}$$

where the transformed displacement operator, which has been used here, reads

$$\hat{D}'(\beta) = \hat{S}(\xi)\hat{D}(\beta)\hat{S}^\dagger(\xi) = \exp(\beta\hat{a}'^\dagger - \beta^*\hat{a}'). \tag{3.123}$$

By means of Eqs (3.113) and (3.114) it can be written in terms of the operators $\hat{a}$ and $\hat{a}^\dagger$ as

$$\hat{D}'(\beta) = \hat{D}(\beta') = \exp(\beta'\hat{a}^\dagger - \beta'^*\hat{a}), \tag{3.124}$$

where the transformed amplitude β' is given by

$$\beta' = \mu\beta - \nu\beta^*. \tag{3.125}$$

Hence the squeezed coherent states can also be obtained by first squeezing the ground state and then displacing it by a modified amplitude β':

$$|\xi,\beta\rangle = \hat{D}(\beta')\hat{S}(\xi)|0\rangle = \hat{D}(\beta')|\xi,0\rangle. \tag{3.126}$$

Taking into consideration that $0 = \hat{U}(\xi,\beta)\hat{a}|0\rangle = \hat{U}(\xi,\beta)\hat{a}\hat{U}^\dagger(\xi,\beta)|\xi,\beta\rangle = \hat{D}'(\beta)\hat{a}'\hat{D}'^\dagger(\beta)|\xi,\beta\rangle = (\hat{a}' - \beta)|\xi,\beta\rangle$ [cf. Eqs (3.47)–(3.49)], we see that the squeezed coherent states are the right-hand eigenstates of the transformed annihilation operator,

$$\hat{a}'|\xi,\beta\rangle = \beta|\xi,\beta\rangle, \tag{3.127}$$

which, in combination with Eq. (3.113), can be regarded as an alternative definition of these states [Yuen (1976)].[2]

Analogously to the coherent states, the squeezed coherent states are overcomplete and nonorthogonal. To prove that they resolve the identity with respect to the coherent amplitude, we use Eq. (3.120) and recall Eq. (3.71):

$$\begin{aligned}\frac{1}{\pi}\int \mathrm{d}^2\beta\, |\xi,\beta\rangle\langle\xi,\beta| &= \hat{S}(\xi)\left(\frac{1}{\pi}\int \mathrm{d}^2\beta\, |\beta\rangle\langle\beta|\right)\hat{S}^\dagger(\xi)\\ &= \hat{S}(\xi)\hat{I}\hat{S}^\dagger(\xi) = \hat{I}.\end{aligned} \tag{3.128}$$

It is also easily seen that the nonorthogonality with respect to different coherent amplitudes but equal squeezing parameters is the same as for the coherent states:

$$\langle\xi,\alpha|\xi,\beta\rangle = \langle\alpha|\hat{S}^\dagger(\xi)\hat{S}(\xi)|\beta\rangle = \langle\alpha|\beta\rangle. \tag{3.129}$$

2) In Yuen's approach to squeezed coherent states (also called two-photon coherent states) the parameter μ is chosen to be complex, μ and ν obeying the condition $|\mu|^2 - |\nu|^2 = 1$. Since here only the phase difference $\arg(\nu) - \arg(\mu)$ is relevant, without loss of generality, $\arg(\mu)$ may be chosen to be zero.

Without going into the details of calculation, we note that the squeeze operator $\hat{S}(\xi)$, Eq. (3.102), can be rewritten, on applying exponential-operator disentangling, as[3]

$$\hat{S}(\xi) = \exp\left(-\frac{\nu}{2\mu}\,\hat{a}^{\dagger 2}\right)\left(\frac{1}{\mu}\right)^{\hat{n}+\frac{1}{2}} \exp\left(\frac{\nu^*}{2\mu}\,\hat{a}^2\right), \tag{3.130}$$

with μ and ν from Eqs (3.115) and (3.116). Hence, the squeezed ground state $|\xi, 0\rangle$, Eq. (3.121), can be given in the form of

$$|\xi, 0\rangle = \frac{1}{\sqrt{\mu}} \exp\left(-\frac{\nu}{2\mu}\,\hat{a}^{\dagger 2}\right)|0\rangle, \tag{3.131}$$

and coherent displacement [according to Eqs (3.122) and (3.124)] then yields the squeezed coherent states in the form of

$$\begin{aligned} |\xi, \beta\rangle &= \frac{1}{\sqrt{\mu}}\,\exp\left[-\frac{\nu}{2\mu}\,(\hat{a}^\dagger - \beta'^*)^2\right]|\beta'\rangle \\ &= \frac{1}{\sqrt{\mu}}\,\exp\left[-\frac{1}{2}\,|\beta'|^2 + \beta'\hat{a}^\dagger - \frac{\nu}{2\mu}\,(\hat{a}^\dagger - \beta'^*)^2\right]|0\rangle, \end{aligned} \tag{3.132}$$

where β' is related to β according to Eq. (3.125). With the help of Eq. (3.132) it is not difficult to prove that the scalar products of the squeezed coherent states $|\xi, \beta\rangle$ with the coherent states $|\alpha\rangle$ and the number states $|n\rangle$, respectively, read[4]

$$\langle \alpha|\xi, \beta\rangle = \frac{1}{\sqrt{\mu}}\,\exp\left(-\frac{|\alpha|^2 + |\beta|^2}{2} + \frac{2\alpha^*\beta - \nu\alpha^{*2} + \nu^*\beta^2}{2\mu}\right), \tag{3.133}$$

$$\langle n|\xi, \beta\rangle = \frac{[\nu/(2\mu)]^{\frac{n}{2}}}{\sqrt{\mu\, n!}}\,\exp\left[-\frac{1}{2}\left(|\beta|^2 - \frac{\nu^*}{\mu}\beta^2\right)\right] \mathrm{H}_n\left(\frac{\beta}{\sqrt{2\mu\nu}}\right), \tag{3.134}$$

with $\mathrm{H}_n(x)$ being the Hermite polynomial.

3.3.1
Statistics of the squeezed states

To obtain the mean number of quanta, or in the case of a radiation-field mode, the mean photon number,

$$\begin{aligned} \langle \xi, \beta|\hat{n}|\xi, \beta\rangle &= \langle \xi, \beta|\hat{a}^\dagger\hat{a}|\xi, \beta\rangle = \langle \beta|\hat{S}^\dagger(\xi)\hat{a}^\dagger\hat{a}\hat{S}(\xi)|\beta\rangle \\ &= \langle \beta|\hat{S}^\dagger(\xi)\hat{a}^\dagger\hat{S}(\xi)\hat{S}^\dagger(\xi)\hat{a}\hat{S}(\xi)|\beta\rangle, \end{aligned} \tag{3.135}$$

3) Equation (3.130) can be proved correct, applying the differential-equation technique described previously in this section and showing that it leads exactly to Eqs (3.113) and (3.114).

4) Note that the relation $\sum_{n=0}^{\infty} \frac{1}{n!}\,\mathrm{H}_n(x)t^n = \exp(-t^2 + 2tx)$ has been used to derive Eq. (3.134).

we first note that $\hat{S}^\dagger(\xi) = \hat{S}(-\xi)$ and that $\mu(-\xi) = \mu(\xi)$ and $\nu(-\xi) = -\nu(\xi)$ [cf. Eqs (3.115) and (3.116)]. From these relations it follows that, on recalling Eq. (3.113),

$$\hat{S}^\dagger(\xi)\hat{a}\hat{S}(\xi) = \hat{S}(-\xi)\hat{a}\hat{S}^\dagger(-\xi) = \mu\hat{a} - \nu\hat{a}^\dagger, \tag{3.136}$$

and we obtain for Eq. (3.135)

$$\begin{aligned}\langle\xi,\beta|\hat{n}|\xi,\beta\rangle &= \langle\beta|(\mu\hat{a}^\dagger - \nu^*\hat{a})(\mu\hat{a} - \nu\,\hat{a}^\dagger)|\beta\rangle \\ &= \langle\beta|(\mu^2\hat{a}^\dagger\hat{a} + |\nu|^2\hat{a}\hat{a}^\dagger - \mu\nu\hat{a}^{\dagger 2} - \mu\nu^*\hat{a}^2)|\beta\rangle \\ &= |\beta'|^2 + |\nu|^2,\end{aligned} \tag{3.137}$$

where $\beta' = \mu\beta - \nu\beta^*$ [Eq. (3.125)]. Analogously, we find for the mean coherent amplitude

$$\langle\xi,\beta|\hat{a}|\xi,\beta\rangle = \langle\beta|\hat{S}^\dagger(\xi)\hat{a}\hat{S}(\xi)|\beta\rangle = \langle\beta|(\mu\,\hat{a} - \nu\,\hat{a}^\dagger)|\beta\rangle = \beta', \tag{3.138}$$

which means that, in the case of a single-mode radiation field, for example, the mean value of the electric-field strength takes the same form as when the mode is in the coherent state $|\beta'\rangle$ [cf. Eq.(3.75)], that is,

$$\langle\xi,\beta|\hat{E}_k(\mathbf{r})|\xi,\beta\rangle = i\omega[A_k(\mathbf{r})\beta' - A_k^*(\mathbf{r})\beta'^*]. \tag{3.139}$$

For $\beta = 0$ (and hence $\beta' = 0$) we see from the last line in Eq. (3.137) that $|\nu|^2$ is the contribution to the mean number of quanta coming from the squeezed ground state $|\xi,0\rangle$,

$$\langle\xi,0|\hat{n}|\xi,0\rangle = |\nu|^2. \tag{3.140}$$

The first term in the last line in Eq. (3.137) obviously constitutes the mean number of quanta coming from the coherent amplitude β' implemented when the squeezed ground state $|\xi,0\rangle$ [Eq. (3.121)] in phase space is displaced by β' to generate the squeezed coherent state $|\xi,\beta\rangle$ [Eq. (3.122)], which is quite similar to the generation of the coherent state $|\beta'\rangle$ from the ordinary ground state $|0\rangle$. Hence $|\beta'|^2$ just corresponds to the mean number of quanta associated with the coherent part of the excitation of the system,

$$\langle\beta'|\hat{n}|\beta'\rangle = |\beta'|^2 = |\langle\xi,\beta|\hat{a}|\xi,\beta\rangle|^2. \tag{3.141}$$

Using Eqs (3.140) and (3.141), we may rewrite Eq. (3.137) as a sum of the mean number of quanta associated with the coherent part and the incoherent part of the excitation, respectively,

$$\langle\xi,\beta|\hat{n}|\xi,\beta\rangle = \langle\beta'|\hat{n}|\beta'\rangle + \langle\xi,0|\hat{n}|\xi,0\rangle. \tag{3.142}$$

For the case of a radiation-field mode, the squeezed ground state $|\xi,0\rangle$ ($\xi\neq 0$) is often called the squeezed vacuum, in contrast to the ordinary vacuum $|0\rangle$, which is the state with zero photons. It should be pointed out that this does not hold for the squeezed vacuum $|\xi,0\rangle$: the mean number of photons in this state does not vanish ($|\nu|^2\neq 0$).

To gain deeper insight into the statistics of squeezed coherent states, it may be useful to introduce the phase-rotated quadrature

$$\hat{x}(\varphi) = \hat{a}\, e^{i\varphi} + \hat{a}^\dagger e^{-i\varphi}, \tag{3.143}$$

which parametrically depends on φ. It can be readily proved that

$$[\hat{x}(\varphi), \hat{x}(\varphi')] = 2i \sin(\varphi - \varphi'). \tag{3.144}$$

Hence, two quadratures of orthogonal phases, i. e., $\varphi' = \varphi \pm \pi/2$, in a similar way to position and momentum, are canonically conjugate to each other, in the sense that

$$[\hat{x}(\varphi), \hat{x}\big(\varphi \pm \tfrac{1}{2}\pi\big)] = \mp 2i. \tag{3.145}$$

The phase-rotated quadrature can be used to represent various physical observables by supplementing a real-valued scaling factor and appropriate choice of the phase φ. For example, identifying $\hat{x}(\varphi)$ with a Cartesian component of the electric-field strength of a single-mode (free) radiation field, we would have

$$\hat{E}_k(\mathbf{r}) = i\omega[A_k(\mathbf{r})\hat{a} - A_k^*(\mathbf{r})\hat{a}^\dagger] = \omega|A_k(\mathbf{r})|\hat{x}(\varphi), \tag{3.146}$$

where the phase is given by

$$\varphi = \arg[A_k(\mathbf{r})] + \tfrac{1}{2}\pi. \tag{3.147}$$

From Eq. (3.138) we immediately obtain the expectation value of the quadrature as

$$\langle \xi,\beta|\hat{x}(\varphi)|\xi,\beta\rangle = \beta' e^{i\varphi} + \beta'^* e^{-i\varphi}, \tag{3.148}$$

which of course corresponds to Eq. (3.139).

Let us now consider the quadrature fluctuation by calculating the variance

$$\begin{aligned}\langle \xi,\beta|[\Delta\hat{x}(\varphi)]^2|\xi,\beta\rangle &= \langle \xi,\beta|\hat{x}^2(\varphi)|\xi,\beta\rangle - [\langle \xi,\beta|\hat{x}(\varphi)|\xi,\beta\rangle]^2. \\ &= \big[\langle \xi,\beta|(\hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger)|\xi,\beta\rangle - 2|\langle \xi,\beta|\hat{a}|\xi,\beta\rangle|^2\big] \\ &\quad + 2\mathrm{Re}\big[\langle \xi,\beta|(\Delta\hat{a})^2|\xi,\beta\rangle\, e^{2i\varphi}\big].\end{aligned} \tag{3.149}$$

Applying Eqs (3.136) and (3.138), we first calculate

$$\begin{aligned}\langle\xi,\beta|(\Delta\hat{a})^2|\xi,\beta\rangle &= \langle\beta|\hat{S}^\dagger(\xi)\hat{a}^2\hat{S}(\xi)|\beta\rangle - \beta'^2 = \langle\beta|(\mu\hat{a}-\nu\hat{a}^\dagger)^2|\beta\rangle - \beta'^2\\ &= \langle\beta|[\mu^2\hat{a}^2+\nu^2\hat{a}^{\dagger 2}-\mu\nu(\hat{a}^\dagger\hat{a}+\hat{a}\hat{a}^\dagger)]|\beta\rangle - \beta'^2\\ &= -\mu\nu,\end{aligned} \tag{3.150}$$

so that, by insertion of Eq. (3.150) into (3.149), we obtain for the sought variance

$$\langle\xi,\beta|[\Delta\hat{x}(\varphi)]^2|\xi,\beta\rangle = \left|\mu e^{i\varphi}-\nu^* e^{-i\varphi}\right|^2. \tag{3.151}$$

Introducing the modulus $|\nu|$ and phase φ_ν $(=\varphi_\xi)$ of ν, we may rewrite Eq. (3.151) as

$$\begin{aligned}\langle\xi,\beta|[\Delta\hat{x}(\varphi)]^2|\xi,\beta\rangle &= \left|\mu-|\nu|\exp[i(2\varphi+\varphi_\nu)]\right|^2\\ &= \left\{1+2|\nu|^2\left[1-\sqrt{\frac{1+|\nu|^2}{|\nu|^2}}\,\cos(2\varphi+\varphi_\nu)\right]\right\},\end{aligned} \tag{3.152}$$

where, in order to express the variance solely in terms of ν, we have used the relation (3.118).

Equation (3.152) reveals that for a fixed value of ν (i.e., fixed ξ) the variance $\langle\xi,\beta|[\Delta\hat{x}(\varphi)]^2|\xi,\beta\rangle$ sensitively depends on the phase $2\varphi+\varphi_\nu$. Clearly, in the limiting case as ν goes to zero (or equivalently $\xi\to 0$), that is, when the squeezed coherent state $|\xi,\beta\rangle$ tends to the ordinary coherent state $|\beta\rangle$, this phase dependence vanishes and we obtain the ground-state quadrature fluctuation:

$$\lim_{\nu\to 0}\langle\xi,\beta|[\Delta\hat{x}(\varphi)]^2|\xi,\beta\rangle = \langle\beta|[\Delta\hat{x}(\varphi)]^2|\beta\rangle = \langle 0|[\Delta\hat{x}(\varphi)]^2|0\rangle = 1. \tag{3.153}$$

For nonvanishing squeezing parameter ($\nu\neq 0$), however, the fluctuation depends, for chosen φ_ν, crucially on φ, so that at certain values of φ the fluctuation may be larger or even smaller than the ground-state limit (3.153). From Eq. (3.152) it can be seen that the fluctuation is smaller than the ground-state limit, i.e., $\langle\xi,\beta|[\Delta\hat{x}(\varphi)]^2|\xi,\beta\rangle<1$, for

$$\cos(2\varphi+\varphi_\nu) > \sqrt{\frac{|\nu|^2}{1+|\nu|^2}}\,, \tag{3.154}$$

and it is minimal for $\cos(2\varphi+\varphi_\nu)=1$. That is, for the specific values of the phase φ given by

$$\varphi_{\min} = k\pi - \tfrac{1}{2}\varphi_\nu \tag{3.155}$$

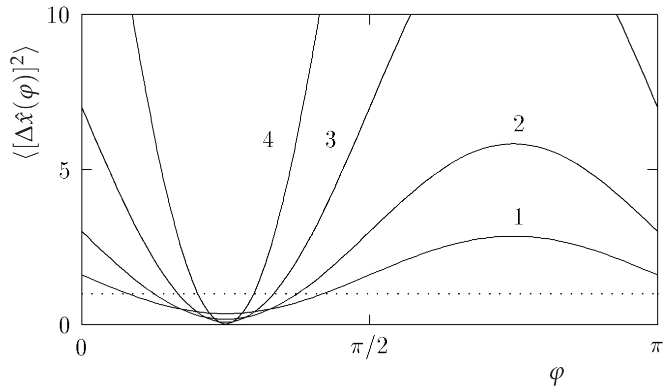

Fig. 3.1 The variance of the quadrature $\langle[\Delta\hat{x}(\varphi)]^2\rangle$ is shown for a squeezed state with $\varphi_\nu = -\pi/2$ and for $|\nu|^2 = 0.3$ (curve 1), 1 (2), 3 (3) and 10 (4); the ground-state noise level ($|\nu|^2 = 0$) is indicated by the dotted line. It can be seen that the noise is π-periodic with respect to the phase φ of the quadrature.

(k, integer), the quadrature fluctuation is reduced below the ground-state limit by the factor

$$\langle\xi,\beta|[\Delta\hat{x}(\varphi)]^2|\xi,\beta\rangle\big|_{\varphi=\varphi_{\min}} = e^{-2|\xi|}. \tag{3.156}$$

On the other hand, the fluctuation becomes larger than the ground-state limit, i. e., $\langle\xi,\beta|[\Delta\hat{x}(\varphi)]^2|\xi,\beta\rangle > 1$, for

$$\cos(2\varphi + \varphi_\nu) < \sqrt{\frac{|\nu|^2}{1+|\nu|^2}}\,. \tag{3.157}$$

Here the maximum fluctuation is observed at phases where $\cos(2\varphi+\varphi_\nu) = -1$, that is for values of φ given by

$$\varphi_{\max} = \tfrac{1}{2}(2k+1)\pi - \tfrac{1}{2}\varphi_\nu, \tag{3.158}$$

where the fluctuation is enhanced with respect to the ground state by the factor

$$\langle\xi,\beta|[\Delta\hat{x}(\varphi)]^2|\xi,\beta\rangle\big|_{\varphi=\varphi_{\max}} = e^{2|\xi|}. \tag{3.159}$$

We see that for certain phase values the quadrature noise can be "squeezed" below the ground-state (vacuum) level at the expense of increased noise for certain other phase values. From inspection of Eq. (3.152), this squeezing effect is seen to increase with $|\nu|$ ($= \cosh|\xi|$). The typical fluctuation behavior of a system in a squeezed coherent state is illustrated in Fig. 3.1. It clearly shows that, with decreasing noise for a given phase $\varphi_{\min}$, the noise for the phase $\varphi_{\max}=\varphi_{\min}+\pi/2$ can drastically increase. Moreover, the more strongly the noise is reduced below the ground-state (vacuum) level, the narrower the phase region around $\varphi_{\min}$, in which noise reduction is observed, becomes.

The behavior is closely related to Heisenberg's uncertainty principle for two observables $\hat{A}$ and $\hat{B}$,

$$\langle(\Delta\hat{A})^2\rangle\langle(\Delta\hat{B})^2\rangle \geq \tfrac{1}{4}|\langle[\hat{A},\hat{B}]\rangle|^2, \tag{3.160}$$

which, because of the commutator relation (3.144) for $\hat{A}=\hat{x}(\varphi)$ and $\hat{B}=\hat{x}(\varphi')$, reads

$$\langle[\Delta\hat{x}(\varphi)]^2\rangle\langle[\Delta\hat{x}(\varphi')]^2\rangle \geq \sin^2(\varphi-\varphi') \tag{3.161}$$

and holds for arbitrary quantum states. In the case when the two phases are $\varphi=\varphi_{\min}$ [Eq. (3.155)] and $\varphi'=\varphi_{\max}$ [Eq. (3.158)] – two phases that correspond to two orthogonal directions in phase space, so that the quadratures $\hat{x}(\varphi_{\min})$ and $\hat{x}(\varphi_{\max})$ are canonically conjugate to each other – then, in agreement with the commutator relation (3.145), the uncertainty relation (3.161) takes the form

$$\langle[\Delta\hat{x}(\varphi_{\min})]^2\rangle\langle[\Delta\hat{x}(\varphi_{\max})]^2\rangle \geq 1. \tag{3.162}$$

For squeezed coherent states, from Eqs (3.156) and (3.159) it follows that

$$\langle\xi,\beta|[\Delta\hat{x}(\varphi_{\min})]^2|\xi,\beta\rangle\langle\xi,\beta|[\Delta\hat{x}(\varphi_{\max})]^2|\xi,\beta\rangle = 1, \tag{3.163}$$

which shows that squeezed coherent states (as also coherent states) are minimum-uncertainty states. It should be pointed out that, in the more general case, where the squeezing phase φ_ν is not necessarily adjusted to the quadrature phases φ and $\varphi+\pi/2$, according to Eqs (3.155) and (3.158), from Eq. (3.151) the uncertainty product

$$\langle\xi,\beta|[\Delta\hat{x}(\varphi)]^2|\xi,\beta\rangle\langle\xi,\beta|\big[\Delta\hat{x}\big(\varphi+\tfrac{1}{2}\pi\big)\big]^2|\xi,\beta\rangle = [1+4\mu^2|\nu|^2\sin^2(2\varphi+\varphi_\nu)] \tag{3.164}$$

follows. Comparing this equation with Eq. (3.163), we find that the squeezed coherent states minimize the uncertainty product for $\hat{x}(\varphi)$ and $\hat{x}(\varphi+\pi/2)$ only when the phase of squeezing is related to φ by

$$2\varphi+\varphi_\nu = k\pi, \tag{3.165}$$

where k is an integer number [Schubert and Vogel (1978a)]. In this case $\langle\xi,\beta|[\Delta\hat{x}(\varphi)]^2|\xi,\beta\rangle$ and $\langle\xi,\beta|[\Delta\hat{x}(\varphi+\pi/2)]^2|\xi,\beta\rangle$ are just the extremal values [Eqs (3.156) and (3.159)]. The coherent states $|\alpha\rangle$, however, are minimum-uncertainty states, independent of the choice of the phase φ:

$$\langle\alpha|[\Delta\hat{x}(\varphi)]^2|\alpha\rangle\langle\alpha|\big[\Delta\hat{x}\big(\varphi+\tfrac{1}{2}\pi\big)\big]^2|\alpha\rangle = 1. \tag{3.166}$$

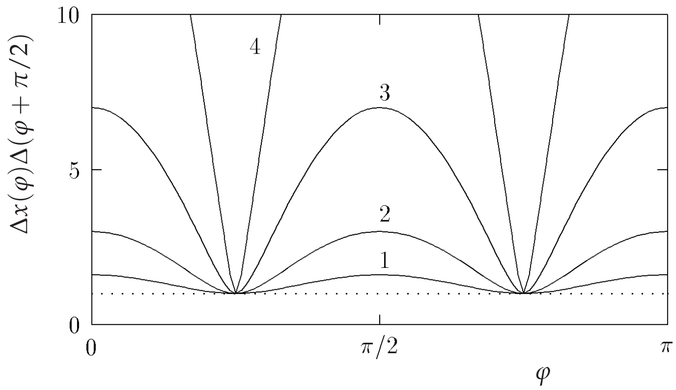

Fig. 3.2 The uncertainty product $\Delta x(\varphi)\Delta x(\varphi+\pi/2) \equiv \{\langle[\Delta\hat{x}(\varphi)]^2\rangle\langle[\Delta\hat{x}(\varphi+\pi/2)]^2\rangle\}^{\frac{1}{2}}$ as a function of φ for a squeezed state with $\varphi_\nu = -\pi/2$ and for $|\nu|^2 = 0.3$ (curve 1), 1 (2), 3 (3) and 10 (4). The case with $|\nu|^2 = 0$ (dotted line) corresponds to a coherent state that minimizes the uncertainty product for all values of φ.

The dependence on the phase φ of the uncertainty product of two canonically conjugated quadratures in the case of squeezed coherent states is illustrated in Fig. 3.2.

Recalling the definition of normally ordered operator products, we can easily prove the relation

$$\langle:[\Delta\hat{x}(\varphi)]^2:\rangle = \langle[\Delta\hat{x}(\varphi)]^2\rangle - \langle 0|[\Delta\hat{x}(\varphi)]^2|0\rangle, \tag{3.167}$$

which is valid for an arbitrary quantum state in which the system is prepared. Since in the case when the system is prepared in a squeezed coherent state $|\xi,\beta\rangle$ the quadrature variance $\langle\xi,\beta|[\Delta\hat{x}(\varphi)]^2|\xi,\beta\rangle$ becomes, for appropriately chosen values of φ, smaller than the vacuum limit $\langle 0|[\Delta\hat{x}(\varphi)]^2|0\rangle$, so that the normally ordered variance $\langle\xi,\beta| : [\Delta\hat{x}(\varphi)]^2 : |\xi,\beta\rangle$ becomes negative. At this point it should be emphasized that the squeezed coherent states may be viewed as a typical but special class of states giving rise to squeezing. Quite general, a quantum state may be said to reveal squeezing if for certain values of the phase the normally ordered quadrature variance becomes negative:

$$\langle:[\Delta\hat{x}(\varphi)]^2:\rangle < 0. \tag{3.168}$$

3.3.2
Multi-mode squeezed states

The criterion for squeezing as given by Eq. (3.168) can also be used to study (complicated) multi-mode systems. Let

$$\hat{X} = \sum_\lambda c_\lambda \hat{x}_\lambda(\varphi_\lambda) \tag{3.169}$$

be the multi-mode quadrature operator. According to Eq. (3.168), squeezing is observed if

$$\langle : (\Delta \hat{X})^2 : \rangle < 0. \tag{3.170}$$

In particular, a multi-mode radiation-field strength

$$\hat{F} = \sum_{\lambda} F_\lambda \hat{a}_\lambda + F_\lambda^* \hat{a}_\lambda^\dagger \tag{3.171}$$

[cf. Eq. (2.271)] can be regarded as a multi-mode quadrature, setting $c_\lambda = |F_\lambda|$ and $\varphi_\lambda = \arg F_\lambda$, and the squeezing criterion reads $\langle : (\Delta \hat{F})^2 : \rangle < 0$, i.e, the normally ordered field variance must be negative.

A generalization of the single-mode squeeze operator defined by Eq. (3.102) to a squeeze operator acting on multi-mode systems is

$$\hat{S} = \exp\left[\sum_{\lambda,\lambda'} \left(\xi^*_{\lambda\lambda'} \hat{a}_\lambda \hat{a}_{\lambda'} - \xi_{\lambda\lambda'} \hat{a}^\dagger_\lambda \hat{a}^\dagger_{\lambda'}\right)\right]. \tag{3.172}$$

If the matrix $\xi_{\lambda\lambda'}$ has only diagonal elements, $\xi_{\lambda\lambda'} = \delta_{\lambda,\lambda'}\xi_\lambda/2$, the multi-mode squeeze operator (3.172) reduces to a product of single-mode squeeze operators of the type (3.102), and its application to the ground state of the multi-mode system generates multi-mode squeezed vacuum states that are simply the (direct) products of single-mode squeezed states. Additional application of the multi-mode displacement operator then generates multi-mode squeezed coherent states, which are of course also direct-product states.

A system is said to genuinely feature multi-mode squeezing, if there are nonvanishing off-diagonal $\xi_{\lambda\lambda'}$ that give rise to nonclassical correlations between the modes. To give a simple but illustrative example, let us consider the case of two modes ($\lambda = 1, 2$) being prepared in a two-mode squeezed vacuum state as a typical example of an entangled state (Section 8.5):

$$|\xi, 0, 0\rangle = \hat{S}(\xi)|0, 0\rangle, \tag{3.173}$$

where $|0,0\rangle \equiv |0_1\rangle|0_2\rangle$ is the ordinary two-mode vacuum state and the two-mode squeeze operator $\hat{S}(\xi)$ is a special case of the general operator (3.172) with $\xi_{12} = \xi$, $\xi_{21} = \xi^*$ and $\xi_{11} = \xi_{22} = 0$,

$$\hat{S}(\xi) = \exp(\xi^* \hat{a}_1 \hat{a}_2 - \xi \hat{a}_1^\dagger \hat{a}_2^\dagger). \tag{3.174}$$

The explicit action of the two-mode squeeze operator on $\hat{a}_1$ and $\hat{a}_2$ can be found in a similar way to that described for the single-mode case. The result is

$$\hat{S}^\dagger(\xi)\hat{a}_1\hat{S}(\xi) = \mu\hat{a}_1 - \nu\hat{a}_2^\dagger, \tag{3.175}$$

$$\hat{S}^\dagger(\xi)\hat{a}_2\hat{S}(\xi) = \mu\hat{a}_2 - \nu\hat{a}_1^\dagger, \tag{3.176}$$

where μ and ν are related to ξ by Eqs (3.115) and (3.116), respectively, and in close analogy to Eq. (3.130), the two-mode squeeze operator can be disentangled to obtain

$$\hat{S} = \exp\left(-\frac{\nu}{\mu}\,\hat{a}_1^\dagger\hat{a}_2^\dagger\right)\left(\frac{1}{\mu}\right)^{\hat{n}_1+\hat{n}_2+1}\exp\left(\frac{\nu^*}{\mu}\,\hat{a}_1\hat{a}_2\right). \tag{3.177}$$

Combining Eqs (3.173) and (3.177), we can easily see that the two-mode squeezed vacuum state can be represented as

$$|\xi,0,0\rangle = \frac{1}{\mu}\,\exp\left(-\frac{\nu}{\mu}\,\hat{a}_1^\dagger\hat{a}_2^\dagger\right)|0,0\rangle = \frac{1}{\mu}\sum_{n=0}^{\infty}\left(-\frac{\nu}{\mu}\right)^n|n,n\rangle. \tag{3.178}$$

The generalization to two-mode squeezed coherent states with nonvanishing coherent amplitudes can be obtained in a straightforward way by additionally applying coherent displacement operators for both modes.

Let us consider a two-mode radiation-field strength

$$\hat{F} = F_1\hat{a}_1 + F_1^*\hat{a}_1^\dagger + F_2\hat{a}_2 + F_2^*\hat{a}_2^\dagger \tag{3.179}$$

$[F_\lambda = |F_\lambda|\exp(i\varphi_\lambda)]$. Taking the two-mode field to be in the squeezed vacuum state $|\xi,0,0\rangle$ as given in Eq. (3.173) and using Eqs (3.175) and (3.176), we calculate the normally ordered variance of $\hat{F}$ to be

$$\begin{aligned}\langle\xi,0,0|:(\Delta\hat{F})^2\!:|\xi,0,0\rangle &= 2\big(|F_1|^2+|F_2|^2\big)|\nu|^2\\ &\times\left[1-\frac{2|F_1F_2|}{|F_1|^2+|F_2|^2}\sqrt{\frac{1+|\nu|^2}{|\nu|^2}}\,\cos(\varphi_1+\varphi_2+\varphi_\nu)\right].\end{aligned} \tag{3.180}$$

We see that the radiation may indeed be squeezed, because for appropriately chosen phases the value of $\langle:(\Delta\hat{F})^2:\rangle$ may become negative.

We identify $\hat{a}_1$ and $\hat{a}_2$ with the annihilation operators for modes of frequencies $\omega_1 = \omega_0 + \Delta\omega/2$ and $\omega_2 = \omega_0 - \Delta\omega/2$, respectively, and apply the model to the calculation of the normally ordered variance of the electric field $\hat{E}(x,t) = \hat{E}^{(+)}(x,t) + \hat{E}^{(-)}(x,t)$,

$$\hat{E}^{(+)}(x,t) = i\int_0^\infty \mathrm{d}\omega\,\sqrt{\frac{\hbar\omega}{4\pi\epsilon_0 c\mathcal{A}}}\,e^{-i\omega(t-x/c)}\hat{a}(\omega), \tag{3.181}$$

of a linearly polarized wave packet propagating in the positive x direction [see Eq. (3.90)]. The quantum state considered here is a squeezed vacuum of the form

$$|\psi\rangle_{\mathrm{sv}} = \hat{S}|\psi\rangle_{\mathrm{v}}, \tag{3.182}$$

where $|\psi\rangle_{\rm v}$ denotes the ordinary vacuum state and the squeeze operator is given as

$$\hat{S} = \exp\left\{ \int_0^\infty \mathrm{d}\omega \, [\xi^*(\omega)\hat{a}(\omega_0+\omega)\hat{a}(\omega_0-\omega) - \xi(\omega)\hat{a}^\dagger(\omega_0+\omega)\hat{a}^\dagger(\omega_0-\omega)] \right\}. \tag{3.183}$$

Such a squeeze operator correlates pairs of modes around the mid-frequency ω_0, quite similar to the previous example given in Eq. (3.174), however, integrated over all difference frequencies. Nevertheless, it can be easily seen that the following transformation holds in close analogy to Eqs (3.175) and (3.176)

$$\hat{S}^\dagger \hat{a}(\omega)\hat{S} = \mu(|\omega-\omega_0|)\hat{a}(\omega) - \nu(|\omega-\omega_0|)\hat{a}^\dagger(2\omega_0-\omega), \tag{3.184}$$

where $\mu(\omega)$ and $\nu(\omega)$ are defined via $\xi(\omega)$ as described in Eqs (3.115) and (3.116), respectively.

From this transformation can immediately be seen that

$$\langle \hat{E}^{(+)}(x,t)\rangle_{\rm sv} = 0. \tag{3.185}$$

Further, by assuming a finite spectral width of squeezing in the sense that $\nu(\omega)\neq 0$ only for $0\leq\omega\leq\Delta\omega$, where the spectral width is small compared with the mid-frequency, $\Delta\omega\ll\omega_0$, it can be shown that the field correlation functions read (to good approximation) as

$$\langle \hat{E}^{(-)}(x,t)\hat{E}^{(+)}(x',t')\rangle_{\rm sv} = \frac{\hbar\omega_0}{2\pi\epsilon_0 c\mathcal{A}} e^{i\omega_0(\tau-\tau')} \int_0^{\Delta\omega} \mathrm{d}\omega \, |\nu(\omega)|^2 \cos[\omega(\tau-\tau')], \tag{3.186}$$

$$\langle \hat{E}^{(+)}(x,t)\hat{E}^{(+)}(x',t')\rangle_{\rm sv} = \frac{\hbar\omega_0}{2\pi\epsilon_0 c\mathcal{A}} e^{-i\omega_0(\tau+\tau')} \int_0^{\Delta\omega} \mathrm{d}\omega \, \mu(\omega)\nu(\omega) \cos[\omega(\tau-\tau')], \tag{3.187}$$

where the notation $\tau=t-x/c$ has been used. Suppose that the times to be resolved are large compared with the inverse bandwidth of the squeezing spectrum, $\Delta\tau\gg(\Delta\omega)^{-1}$, and that the squeezing spectrum is sufficiently flat, i. e., $\nu(\omega)\simeq\bar{\nu}$ and $\mu(\omega)\simeq\bar{\mu}$. In this case the frequency integrals in Eqs (3.186) and (3.187) can be approximated as delta functions and we obtain:

$$\langle \hat{E}^{(-)}(x,t)\hat{E}^{(+)}(x',t')\rangle_{\rm sv} = \frac{\hbar\omega_0}{2\epsilon_0 c\mathcal{A}} |\bar{\nu}|^2 \exp[i\omega_0(\tau-\tau')]\delta(\tau-\tau'), \tag{3.188}$$

$$\langle \hat{E}^{(+)}(x,t)\hat{E}^{(+)}(x',t')\rangle_{\rm sv} = \frac{\hbar\omega_0}{2\epsilon_0 c\mathcal{A}} \bar{\mu}\bar{\nu} \exp[-i\omega_0(\tau+\tau')]\delta(\tau-\tau'). \tag{3.189}$$

A radiation field with mean value and correlation functions according to Eqs (3.185), (3.188) and (3.189) is usually called squeezed white noise [Gardiner (1991)]. Note that for an ordinary white-noise field, the relations $\langle\hat{E}^{(+)}\hat{E}^{(+)}\rangle = \langle\hat{E}^{(-)}\hat{E}^{(-)}\rangle = 0$ hold.

3.4
Quadrature eigenstates

So far we have studied the eigenstates of various kinds of operators, such as Hermitian number operators $\hat{n} = \hat{a}^\dagger\hat{a}$ (number states), non-Hermitian photon destruction operators $\hat{a}$ (coherent states), and linear combinations of photon destruction and creation operators $\mu\hat{a} + \nu\hat{a}^\dagger$ (squeezed coherent states). In this context, the question arises as to what are the eigenstates of the Hermitian phase-rotated quadrature operator $\hat{x}(\varphi)$ defined by Eq. (3.143).

Before going into detail and answering the question we first mention the limiting properties of the squeezed coherent states. For this purpose let us consider the commutator of the annihilation operator $\hat{a}' = \mu\hat{a} + \nu\hat{a}^\dagger$ and the quadrature operator,

$$\begin{aligned} [\hat{x}(\varphi), \hat{a}'] &= [\hat{a}e^{i\varphi} + \hat{a}^\dagger e^{-i\varphi}, \mu\hat{a} + \nu\hat{a}^\dagger] \\ &= |\nu|e^{-i\varphi}\left[e^{i(2\varphi+\varphi_\nu)} - \sqrt{\frac{1+|\nu|^2}{|\nu|^2}}\right]. \end{aligned} \tag{3.190}$$

Choosing the phases as $2\varphi + \varphi_\nu = 2\pi k$ where k is an integer number, the commutator vanishes in the limit of infinite squeezing:

$$\lim_{|\nu|\to\infty} [\hat{x}(\varphi), \hat{a}'] = 0 \qquad (2\varphi + \varphi_\nu = 2\pi k). \tag{3.191}$$

That is, in the considered limit the operators $\hat{x}(\varphi)$ and $\hat{a}'$ obviously have the same eigenstates. In the limit as $|\nu| \to \infty$ the eigenstates of $\hat{a}'$ represent ideally squeezed coherent states, so that (for appropriately chosen phase φ) the eigenstates of the quadrature operator can be viewed as ideally squeezed coherent states in that limit.

We now turn to the problem of deriving the explicit form of the (single-mode) quadrature eigenstates in terms of number states.[5] For this purpose let us first consider the case $\varphi = 0$, for which the form of $\hat{x}(\varphi)$ corresponds to the well known position operator. The corresponding eigenvalue equation expanded in terms of number states reads

$$\sqrt{n+1}\,\langle n+1|x\rangle + \sqrt{n}\,\langle n-1|x\rangle = x\langle n|x\rangle. \tag{3.192}$$

5) Alternative derivations of the phase-rotated quadrature eigenstates in both number-state and coherent-state representations were given by Schubert and Vogel (1978b).

The normalized solution of Eq. (3.192) can be written in terms of the Hermite polynomials as

$$\langle n|x\rangle = \psi_n(x) = \left(2^n n!\sqrt{2\pi}\right)^{-\frac{1}{2}} \mathrm{H}_n\left(x/\sqrt{2}\right)e^{-\frac{1}{4}x^2}. \tag{3.193}$$

For an arbitrary phase φ we employ the phase-rotation operator given by

$$\hat{U}(\varphi) = \exp(-i\varphi\hat{a}^\dagger\hat{a}), \tag{3.194}$$

with the help of which it may be easily proved that

$$\hat{x}(0) = \hat{U}^\dagger(\varphi)\hat{x}(\varphi)\hat{U}(\varphi). \tag{3.195}$$

The general eigenvalue problem can then be written as

$$\hat{x}(0)|x\rangle = \hat{U}^\dagger(\varphi)\hat{x}(\varphi)\hat{U}(\varphi)|x\rangle = x|x\rangle, \tag{3.196}$$

or, multiplying from the left-hand side by $\hat{U}(\varphi)$,

$$\hat{x}(\varphi)\hat{U}(\varphi)|x\rangle = x\hat{U}(\varphi)|x\rangle. \tag{3.197}$$

This shows that the eigenstates of $\hat{x}(\varphi)$ are simply given by

$$|x,\varphi\rangle = \hat{U}(\varphi)|x\rangle, \tag{3.198}$$

which in the number basis reads as, on recalling Eq. (3.193),

$$\langle n|x,\varphi\rangle = \psi_n(x)e^{-in\varphi}. \tag{3.199}$$

Clearly, the quadrature eigenstates $|x,\varphi\rangle$ can also be expressed in terms other than number states. In particular, in the case of a coherent-state representation we have

$$|x,\varphi\rangle = \frac{1}{\pi}\int \mathrm{d}^2\alpha\, |\alpha\rangle\langle\alpha|x,\varphi\rangle. \tag{3.200}$$

The scalar product $\langle\alpha|x,\varphi\rangle$ can be obtained in different ways, one of which is by use of the number states and direct evaluation of the occurring sums (cf. footnote 4, p. 92). The result is

$$\begin{aligned}\langle\alpha|x,\varphi\rangle = {} & (2\pi)^{-\frac{1}{4}}\exp\left[-\tfrac{1}{4}x^2 + x|\alpha|e^{-i(\varphi+\varphi_\alpha)}\right] \\ & \times\, \exp\left\{-|\alpha|^2\cos^2(\varphi+\varphi_\alpha) + \tfrac{1}{2}i|\alpha|^2\sin[2(\varphi+\varphi_\alpha)]\right\},\end{aligned} \tag{3.201}$$

where $\varphi_\alpha = \arg(\alpha)$. Equation (3.201) reveals that the probability distribution for observing a value of the quadrature x when the system is prepared in a coherent state $|\alpha\rangle$ is a Gaussian,

$$|\langle\alpha|x,\varphi\rangle|^2 = \frac{1}{\sqrt{2\pi}}\exp\left\{-\tfrac{1}{2}[x-\langle\hat{x}(\varphi)\rangle]^2\right\}, \tag{3.202}$$

where the mean value reads

$$\langle \hat{x}(\varphi) \rangle = \langle \alpha | \hat{x}(\varphi) | \alpha \rangle = 2|\alpha| \cos(\varphi + \varphi_\alpha). \tag{3.203}$$

We finally note that, for given φ, the quadrature eigenstates are of course orthogonal and complete in the sense that

$$\langle x, \varphi | x', \varphi \rangle = \delta(x - x'), \tag{3.204}$$

and

$$\int \mathrm{d}x \, |x, \varphi\rangle \langle x, \varphi| = \hat{I}, \tag{3.205}$$

which can be proved by using the explicit form of the quadrature eigenstates in the number-state or coherent-state representation as given above.

3.5 Phase states

As we know, the annihilation and creation operators $\hat{a}$ and $\hat{a}^\dagger$ correspond to the classical complex amplitudes α and α^*, respectively, according to the relations

$$\hat{a} \mapsto \alpha = |\alpha| e^{i\varphi}, \tag{3.206}$$

$$\hat{a}^\dagger \mapsto \alpha^* = |\alpha| e^{-i\varphi}. \tag{3.207}$$

Thus in classical physics it is straightforward to express the quantities of the system in terms of the amplitude and phase variables $|\alpha|$ and φ, respectively. Amplitude and phase seem to appear in this context as observable quantities and one may therefore ask for the quantum-mechanical operators which, in a sense, correspond to them. Attempts to introduce amplitude and phase variables in quantum mechanics are nearly as old as quantum mechanics itself. Since Dirac's introduction of amplitude and phase operators in 1927, a series of concepts have been developed. Here we concentrate on only a few of them and emphasize more the resulting phase states that are eigenstates of appropriately chosen phase operators.

In close analogy with the classical approach to the problem of defining amplitude and phase variables, Dirac introduced a phase operator $\hat{\phi}$ by factoring the annihilation and creation operators as follows:

$$\hat{a} = \hat{V} \sqrt{\hat{n}}\,, \quad \hat{a}^\dagger = \sqrt{\hat{n}}\, \hat{V}^\dagger, \tag{3.208}$$

where the operator $\hat{V}$ is regarded as being a unitary operator of the form

$$\hat{V} = e^{i\hat{\phi}} \tag{3.209}$$

[Dirac (1927)], with $\hat{\phi}$ being assumed to be the Hermitian phase operator,

$$\hat{\phi}^{\dagger} = \hat{\phi}. \tag{3.210}$$

However, difficulties arise here which are closely related to the fact that the operator $\hat{V}$ is actually not unitary [London (1926, 1927)] and therefore Eqs (3.208) and (3.209) do not define a Hermitian phase operator $\hat{\phi}$. It can be proved [Carruthers and Nieto (1968)] that $\langle 0|\hat{V}^{\dagger}\hat{V}|0\rangle = 0$, which contradicts the assumption of unitarity, $\hat{V}^{\dagger}\hat{V} = \hat{I}$. This result can be shown as follows. Applying Eq. (3.208) to a photon-number state $|n\rangle$ yields

$$\hat{V}|n\rangle = |n-1\rangle, \qquad n = 1, 2, \ldots. \tag{3.211}$$

In the case $n = 0$, due to the completeness of the number states, one may write

$$\hat{V}|0\rangle = \sum_{n=0}^{\infty} d_n |n\rangle. \tag{3.212}$$

From these equations one finds that

$$\hat{V}^{\dagger}|n\rangle = \sum_{m} |m\rangle\langle m|\hat{V}^{\dagger}|n\rangle = d_n^{*}|0\rangle + |n+1\rangle, \tag{3.213}$$

and hence for $n > 0$

$$\hat{V}^{\dagger}\hat{V}|n\rangle = \hat{V}^{\dagger}|n-1\rangle = d_{n-1}^{*}|0\rangle + |n\rangle. \tag{3.214}$$

Therefore, if $\hat{V}^{\dagger}\hat{V} = \hat{I}$ it follows that $d_n = 0$ for all n. This means that $\hat{V}|0\rangle = 0$ and therefore $\langle 0|\hat{V}^{\dagger}\hat{V}|0\rangle = 0$. Note that in contrast to $\hat{V}^{\dagger}\hat{V}$, $\hat{V}\hat{V}^{\dagger}$ is the identity operator: $\hat{V}\hat{V}^{\dagger} = \hat{I}$.

3.5.1 The eigenvalue problem of $\hat{V}$

Susskind and Glogower (1964) considered, according to Eq. (3.208), the exponential phase operator $\hat{V}$ but without assuming its unitarity:

$$\hat{V} = \widehat{e^{i\phi}}, \qquad \hat{V}^{\dagger} = \left(\widehat{e^{i\phi}}\right)^{\dagger} \tag{3.215}$$

[see also Carruthers and Nieto (1968)]. To represent the operators $\hat{V}$ and $\hat{V}^{\dagger}$ in the number basis, we use the number representations of $\sqrt{\hat{n}}$, $\hat{a}$ and $\hat{a}^{\dagger}$, namely

$$\sqrt{\hat{n}} = \sum_{n=0}^{\infty} \sqrt{n}\,|n\rangle\langle n| = \sum_{n=0}^{\infty} \sqrt{n+1}\,|n+1\rangle\langle n+1|, \tag{3.216}$$

$$\hat{a} = \sum_{n=0}^{\infty} \sqrt{n+1}\,|n\rangle\langle n+1|, \tag{3.217}$$

$$\hat{a}^{\dagger} = \sum_{n=0}^{\infty} \sqrt{n+1}\,|n+1\rangle\langle n| \tag{3.218}$$

[cf. Eqs (3.19) and (3.20)]. Combining Eqs (3.217) and (3.216) and taking into account the completeness of the number states, we derive

$$\hat{a} = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sqrt{n+1}\, |n\rangle\langle n+1|m\rangle\langle m| = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \sqrt{m}\, |n\rangle\langle n+1|m\rangle\langle m| = \sum_{n=0}^{\infty} |n\rangle\langle n+1|\sqrt{\hat{n}}\,, \tag{3.219}$$

from which, together with Eq. (3.208), we may choose $\hat{V}$ as

$$\hat{V} = \sum_{n=0}^{\infty} |n\rangle\langle n+1|, \tag{3.220}$$

and the relation

$$\hat{V}\sqrt{\hat{n}} = \sqrt{\hat{n}+1}\,\hat{V} \tag{3.221}$$

holds.[6] Accordingly, we have

$$\hat{V}^\dagger = \sum_{n=0}^{\infty} |n+1\rangle\langle n|. \tag{3.222}$$

Applying $\hat{V}$ to a number state $|n\rangle$ gives, on using Eq. (3.220),

$$\hat{V}|n\rangle = |n-1\rangle. \tag{3.223}$$

In particular, the application of $\hat{V}$ to the ground state gives

$$\hat{V}|0\rangle = 0. \tag{3.224}$$

Combining Eqs (3.220) and (3.222), we derive

$$\hat{V}\hat{V}^\dagger = \hat{I}, \tag{3.225}$$

$$\hat{V}^\dagger\hat{V} = \hat{I} - |0\rangle\langle 0|. \tag{3.226}$$

Equations (3.225) and (3.226) imply that

$$[\hat{V}, \hat{V}^\dagger] = |0\rangle\langle 0|. \tag{3.227}$$

It should be noted that the nonunitarity and the noncommuting nature are only relevant for states $|\Psi\rangle$ having a significant overlap with the ground state (vacuum):

$$\langle\Psi|[\hat{V}, \hat{V}^\dagger]|\Psi\rangle = |\langle 0|\Psi\rangle|^2. \tag{3.228}$$

6) Note that there is an ambiguity because of the undetermined term with $m=0$ in the second line of Eq. (3.219). From the first line of Eq. (3.219) an ambiguous definition of $\hat{V}$ can be given by supposing that $\hat{a}=(\hat{n}+1)^{1/2}\hat{V}$, which implies that $\hat{V}\hat{n}^{1/2}=(\hat{n}+1)^{1/2}\hat{V}$.

Let us now consider the eigenvalue problem for $\hat{V}$. Postulating

$$\hat{V}|\phi\rangle = e^{i\phi}|\phi\rangle \tag{3.229}$$

and expanding $|\phi\rangle$ in the number basis,

$$|\phi\rangle = \sum_{n=0}^{\infty} b_n|n\rangle, \tag{3.230}$$

we readily arrive, on using Eq. (3.220), at the recurrence relation

$$b_{n+1} = e^{i\phi}b_n, \tag{3.231}$$

which may be satisfied by choosing

$$b_n = b_0 e^{in\phi}. \tag{3.232}$$

The eigenstates of the operator $\hat{V}$ have therefore the form

$$|\phi\rangle = b_0 \sum_{n=0}^{\infty} e^{in\phi}|n\rangle. \tag{3.233}$$

As expected, application of $\hat{U}(\varphi)$, Eq. (3.194), onto $|\phi\rangle$ shifts ϕ to $\phi-\varphi$,

$$\hat{U}(\varphi)|\phi\rangle = |\phi - \varphi\rangle. \tag{3.234}$$

Furthermore, the states $|\phi\rangle$ obviously satisfy the periodicity condition

$$|\phi + 2\pi\rangle = |\phi\rangle, \tag{3.235}$$

and the identity can be resolved by these states:

$$\begin{aligned} \left|b_0\sqrt{2\pi}\right|^{-2} \int_0^{2\pi} \mathrm{d}\phi\, |\phi\rangle\langle\phi| &= \sum_{n,m=0}^{\infty} |n\rangle\langle m|\, \frac{1}{2\pi} \int_0^{2\pi} \mathrm{d}\phi\, \exp[i(n-m)\phi] \\ &= \sum_{n,m=0}^{\infty} |n\rangle\langle m|\, \delta_{nm} = \hat{I}, \end{aligned} \tag{3.236}$$

from which we may choose $b_0 = 1/\sqrt{2\pi}$.

Having in mind a classical picture of phase, the states

$$|\phi\rangle = \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} e^{in\phi}|n\rangle, \tag{3.237}$$

may be regarded as quantum-mechanical phase states [cf. London (1926, 1927)]. The thus introduced phase is also called the canonical phase or London phase. In particular, Eq. (3.234) implies that a freely evolving phase state

($\varphi \mapsto \omega t$) would remain a phase state for all time. However, the states $|\phi\rangle$ are not orthogonal and, unfortunately, they cannot be normalized in a proper way. Indeed, we deduce from Eq. (3.237) that[7]

$$\begin{aligned}\langle\phi|\phi'\rangle &= \frac{1}{2\pi}\sum_{n=0}^{\infty}\exp[-in(\phi-\phi')] \\ &= \left\{\frac{1}{4\pi}+\frac{1}{2}\,\delta(\phi-\phi')-\frac{i}{4\pi}\,\cot\left[\tfrac{1}{2}(\phi-\phi')\right]\right\}\end{aligned} \tag{3.238}$$

($0\le|\phi-\phi'|<2\pi$). Clearly, the states $|\phi\rangle$ are not eigenstates of the operator $\hat{V}^\dagger$, which is significant for an analysis of states which substantially overlap with the ground state. Combining Eqs (3.222) and (3.237) we find that

$$\hat{V}^\dagger|\phi\rangle = e^{-i\phi}\left(|\phi\rangle - \frac{1}{\sqrt{2\pi}}\,|0\rangle\right). \tag{3.239}$$

Nevertheless, the states $|\phi\rangle$ may be useful because they resolve the identity, Eq. (3.236). Hence any state $|\Psi\rangle$ can be expressed in terms of them:

$$|\Psi\rangle = \int_0^{2\pi} \mathrm{d}\phi\,|\phi\rangle\langle\phi|\Psi\rangle. \tag{3.240}$$

The nonorthogonality of the states $|\phi\rangle$ might be removed by using a finite-dimensional Hilbert space spanned by $r+1$ number states $\{|n\rangle\}$. In this truncated Hilbert space, a set of $r+1$ phase states $|\phi^{(r)}\rangle$ can be introduced as

$$|\phi_m^{(r)}\rangle = \frac{1}{\sqrt{r+1}}\sum_{n=0}^{r} e^{in\phi_m^{(r)}}|n\rangle, \tag{3.241}$$

where

$$\phi_m^{(r)} = \phi_0^{(r)} + \frac{2m\pi}{r+1} \qquad (m=0,1,\ldots,r). \tag{3.242}$$

[cf. Eq. (3.237)]. Here the phase $\phi_0^{(r)}$ is a reference phase whose value determines the choice of the 2π periodicity interval of the phase. For each finite r the states $|\phi_m^{(r)}\rangle$ are orthonormal and complete in the sense that

$$\langle\phi_m^{(r)}|\phi_{m'}^{(r)}\rangle = \delta_{mm'}, \tag{3.243}$$

$$\sum_{m=0}^{r}|\phi_m^{(r)}\rangle\langle\phi_m^{(r)}| = \hat{I}. \tag{3.244}$$

7) Note that the relations $2\sum_{n=1}^{\infty}\sin(n\phi)=\cot(\phi/2)$ and $\sum_{n=1}^{\infty}\cos(n\phi)$ $-1/2=\pi\sum_{n=-\infty}^{\infty}\delta(\phi-2n\pi)$ are valid, the latter results from the identity $\sum_{n=-\infty}^{\infty}\exp(in\phi)=2\pi\sum_{n=-\infty}^{\infty}\delta(\phi-2n\pi)$.

Hence in the truncated Hilbert space a Hermitian phase operator can be defined as follows:

$$\hat{\phi}^{(r)} = \sum_{m=0}^{r} \phi_m^{(r)} |\phi_m^{(r)}\rangle\langle\phi_m^{(r)}|. \tag{3.245}$$

The limiting procedure $r \to \infty$ can then be performed at the end of all (c-number) calculations [Loudon (1973); Pegg and Barnett (1988, 1989); Barnett and Pegg (1992)].

3.5.2
Cosine and sine phase states

It is worth noting that the operators $\hat{V}$ and $\hat{V}^\dagger$ may be used to define Hermitian operator analogues of $\cos\phi$ and $\sin\phi$ as follows:

$$\hat{C} = \tfrac{1}{2}(\hat{V} + \hat{V}^\dagger), \tag{3.246}$$

$$\hat{S} = \tfrac{1}{2}i(\hat{V} - \hat{V}^\dagger). \tag{3.247}$$

Using Eqs (3.220) and (3.222), we see that

$$[\hat{V}, \hat{n}] = \hat{V}, \qquad [\hat{V}^\dagger, \hat{n}] = -\hat{V}^\dagger. \tag{3.248}$$

These commutation rules imply, on using Eqs (3.246) and (3.247), the following commutation rules for $\hat{C}$, $\hat{S}$ and $\hat{n}$:

$$[\hat{C}, \hat{n}] = i\hat{S}, \quad [\hat{S}, \hat{n}] = -i\hat{C}, \quad [\hat{C}, \hat{S}] = \tfrac{1}{2}i\hat{P}_0\,, \tag{3.249}$$

where $\hat{P}_0 = |0\rangle\langle 0|$. Hence according to Heisenberg's uncertainty principle (3.160), the following uncertainty relations can be deduced:

$$\Delta n \Delta C \geq \tfrac{1}{2}\langle\hat{S}\rangle, \quad \Delta n \Delta S \geq \tfrac{1}{2}\langle\hat{C}\rangle, \quad \Delta S \Delta C \geq \tfrac{1}{4}\langle\hat{P}_0\rangle. \tag{3.250}$$

In particular, the third of these reveals that C and S can be accurately measured simultaneously only when the state, say $|\Psi\rangle$, has sufficiently small overlap with the ground state: $|\langle 0|\Psi\rangle|^2 \ll 1$. In other words, if the overlap cannot be disregarded, $\hat{C}$ and $\hat{S}$ are expected to give rise to two (Hermitian) phase operators $\hat{\phi}_\mathrm{C}$ and $\hat{\phi}_\mathrm{S}$ instead of the desired one-phase operator. Since $\hat{C}$ and $\hat{S}$ are well-defined Hermitian operators, their eigenvalues give possible results of measurements of C and S.

In order to solve the eigenvalue problem for $\hat{C}$,

$$\hat{C}|\cos\phi\rangle = C|\cos\phi\rangle, \tag{3.251}$$

we expand $|\cos\phi\rangle$ in the number basis,

$$|\cos\phi\rangle = \sum_{n=0}^{\infty} b_n |n\rangle. \tag{3.252}$$

Recalling Eqs (3.220) and (3.222), after some algebra we obtain the recurrence relations

$$2b_0 C = b_1\,, \quad 2b_{n+1} C = b_n + b_{n+2}\,. \tag{3.253}$$

The second of these is solved by

$$b_n = \alpha V^n + \beta V^{-n}, \quad C = \tfrac{1}{2}\,(V + V^{-1}) \tag{3.254}$$

for arbitrary values of α and β. To avoid divergence difficulties $|V|$ must be unity so that

$$V = e^{i\phi}, \quad C = \cos\phi. \tag{3.255}$$

To specify α and β, we note that b_0 can be chosen to be real so that $\alpha = \beta^*$ in Eq. (3.254). Making the substitution $b_0 \mapsto b_0 \sin\phi$ (b_0 real), it is seen from Eqs (3.254) and (3.255) that

$$b_n = b_0 \sin[(n+1)\phi]. \tag{3.256}$$

Hence Eqs (3.251) and (3.252) become

$$\hat{C}|\cos\phi\rangle = \cos\phi|\cos\phi\rangle, \tag{3.257}$$

$$|\cos\phi\rangle = b_0 \sum_{n=0}^{\infty} \sin[(n+1)\phi]|n\rangle. \tag{3.258}$$

Note that all independent solutions are contained in the interval $0 \le \phi < \pi$. By straightforward calculation, on recalling the formulae in footnote 7, p. 108, it can be shown that the $|\cos\phi\rangle$ form an orthonormal and complete set of basis vectors in the Hilbert space ($b_0 = \sqrt{2/\pi}$):

$$\langle\cos\phi|\cos\phi'\rangle = \delta(\phi - \phi'), \tag{3.259}$$

$$\int_0^{\pi} \mathrm{d}\phi\,|\cos\phi\rangle\langle\cos\phi| = \hat{I}. \tag{3.260}$$

The solution of the eigenvalue problem for the sine operator $\hat{S}$ may be found in a very similar way [for details see Carruthers and Nieto (1968)]. The result may be written as

$$\hat{S}|\sin\phi\rangle = \sin\phi|\sin\phi\rangle, \tag{3.261}$$

$$|\sin\phi\rangle = \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} \{\exp[i(n+1)\phi] - \exp[-i(n+1)(\phi - \pi)]\}|n\rangle, \tag{3.262}$$

$$\langle\sin\phi|\sin\phi'\rangle = \delta(\phi - \phi'), \tag{3.263}$$

$$\int_{-\pi/2}^{\pi/2} \mathrm{d}\phi\,|\sin\phi\rangle\langle\sin\phi| = \hat{I}. \tag{3.264}$$

By power-series expansion, for each operator $\hat{C}$ and $\hat{S}$, Hermitian phase operators $\hat{\phi}_{\mathrm{C}}$ $(=\hat{\phi}_{\mathrm{C}}^{\dagger})$ and $\hat{\phi}_{\mathrm{S}}$ $(=\hat{\phi}_{\mathrm{S}}^{\dagger})$, respectively, can be defined as

$$\hat{\phi}_{\mathrm{C}} \equiv \cos^{-1}\hat{C} = \tfrac{1}{2}\pi - \sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}\binom{-\frac{1}{2}}{k}\hat{C}^{2k+1}, \tag{3.265}$$

$$\hat{\phi}_{\mathrm{S}} \equiv \sin^{-1}\hat{S} = \sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}\binom{-\frac{1}{2}}{k}\hat{S}^{2k+1}. \tag{3.266}$$

Note that $\hat{\phi}_{\mathrm{C}}$ and $\hat{\phi}_{\mathrm{S}}$ do not commute ($[\hat{\phi}_{\mathrm{C}}, \hat{\phi}_{\mathrm{S}}] \neq 0$). Hence two unitary operators $\hat{V}_C$ and $\hat{V}_S$ can be defined:

$$\hat{V}_C = e^{i\hat{\phi}_{\mathrm{C}}} \qquad (\hat{V}_C^{\dagger}\hat{V}_C = \hat{V}_C\hat{V}_C^{\dagger} = \hat{I}), \tag{3.267}$$

$$\hat{V}_S = e^{i\hat{\phi}_{\mathrm{S}}} \qquad (\hat{V}_S^{\dagger}\hat{V}_S = \hat{V}_S\hat{V}_S^{\dagger} = \hat{I}), \tag{3.268}$$

so that, combining Eqs (3.246) and (3.247) with the inverse of Eqs (3.265) and (3.266),

$$\hat{V} = \tfrac{1}{2}\left(e^{i\hat{\phi}_{\mathrm{C}}} + e^{i\hat{\phi}_{\mathrm{S}}} + e^{-i\hat{\phi}_{\mathrm{C}}} - e^{-i\hat{\phi}_{\mathrm{S}}}\right). \tag{3.269}$$

This result together with Eq. (3.208) is the correct version of Dirac's postulate given in Eq. (3.209).

The cosine and sine operators of the phases of two modes can be used to define cosine and sine operators of the phase difference, by applying the addition theorems as follows:

$$\hat{C}_{12} = \hat{C}_1\hat{C}_2 + \hat{S}_1\hat{S}_2\,, \tag{3.270}$$

$$\hat{S}_{12} = \hat{S}_1\hat{C}_2 - \hat{S}_2\hat{C}_1\,. \tag{3.271}$$

From the commutation relations (3.249) it is easily shown that $\hat{C}_{12}$ and $\hat{S}_{12}$ commute with the total-number operator:

$$[\hat{C}_{12}, \hat{n}_1 + \hat{n}_2] = 0, \quad [\hat{S}_{12}, \hat{n}_1 + \hat{n}_2] = 0 \tag{3.272}$$

(note that operators of different modes commute).

References

Adam, P., I. Földesi, and J. Janszky (1994) *Phys. Rev. A* **49**, 1281.

Barnett, S.M. and D.T. Pegg (1992) *J. Mod. Opt.* **39**, 1221.

Carruthers, P. and M.M. Nieto (1968) *Rev. Mod. Phys.* **40**, 411.

Dirac, P.A.M. (1927) *Proc. R. Soc. Lond. A* **114**, 243.

Gardiner, C.W. (1991) *Quantum Noise* (Springer Verlag, Berlin).

Glauber, R.J. (1963a) *Phys. Rev.* **130**, 2529.

Glauber, R.J. (1963b) *Phys. Rev.* **131**, 2766.

Glauber, R.J. (1963c) *Phys. Rev. Lett.* **10**, 84.

Klauder, J.R. (1960) *Ann. Phys.* **11**, 123.

London, F. (1926) *Z. Phys.* **37**, 915.

London, F. (1927) *Z. Phys.* **40**, 193.

Loudon, R. (1973) *The Quantum Theory of Light* (Oxford University Press, Oxford).

Loudon, R. and P.L. Knight (1987) *J. Mod. Opt.* **34**, 709.

Mandel, L. and E. Wolf (1995) *Optical Coherence and Quantum Optics* (Cambridge University Press, Cambridge).

Pegg, D.T. and S.M. Barnett (1988) *Europhys. Lett.* **6**, 483.

Pegg, D.T. and S.M. Barnett (1989) *Phys. Rev. A* **39**, 1665.

Peřina, J. (1985) *Coherence of Light* (Reidel, Dordrecht).

Schrödinger, E. (1926) *Naturwiss.* **14**, 664.

Schubert, M. and W. Vogel (1978a) *Phys. Lett. A* **68**, 321.

Schubert, M. and W. Vogel (1978b) *Wiss. Z. Univ. Jena, Math.-Naturwiss. Reihe* **27**, 179.

Stoler, D. (1970) *Phys. Rev. D* **1**, 3217.

Stoler, D. (1971) *Phys. Rev. D* **4**, 1925.

Susskind, L. and J. Glogower (1964) *Physics* **1**, 49.

Yuen, H.P. (1976) *Phys. Rev. A* **13**, 2226.

Walls, D.F. (1983) *Nature* **306**, 141.